Pulse Digital Modulation

Communication is the process of establishing connection or link between two points for information exchange. (or)

Communication is simply the process of conveying message at a distance or communication is the basic process of exchanging information.

→ Depending on message signal communication is classified into two types. They are

1. Analog communication.
2. Digital communication.

Analog communication:-

In Analog communication the modulating signal is Analog in nature. The modulating signal is transmitted with the help of transmitting antenna. At the receiver the signal is received and processed to recover original message signal. The block diagram of analog communication system is shown below.

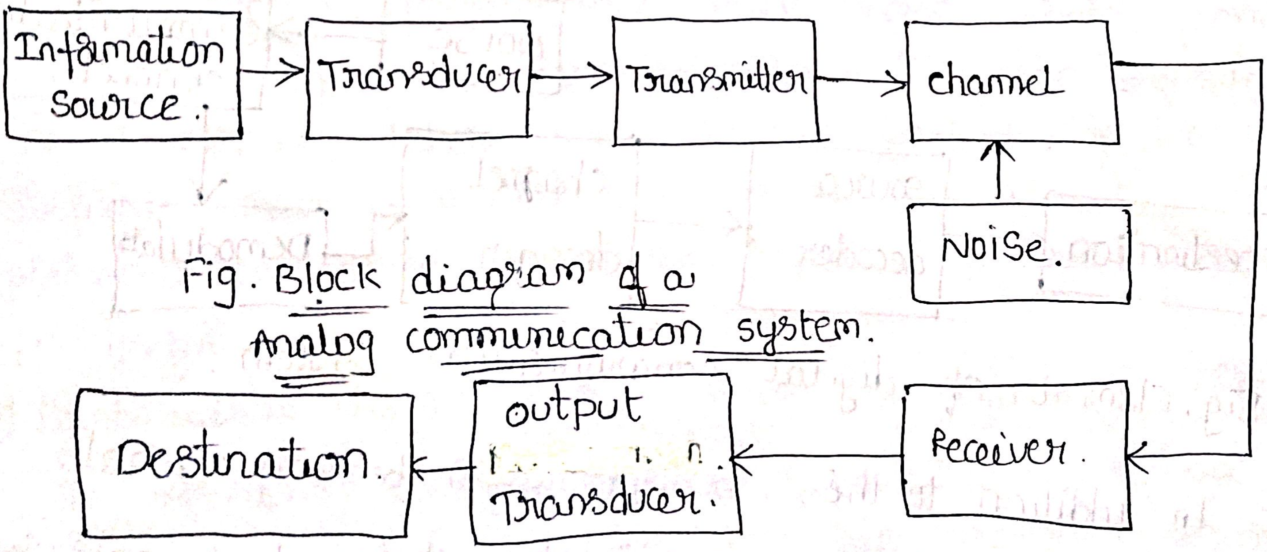


Fig. Block diagram of a Analog communication system.

Presently all the AM, FM radio transmission and TV transmissions are examples of analog communications.

Digital communications

In digital communication the modulating signal is digital in nature.

Elements of digital communication system:-

The below figure elements of digital communication system. The overall purpose of the system is to transmit the message coming out of a source to a destination point at as high rate and accuracy as possible. The communication channels accepts electrical signals and the output of the channel is usually distorted version of the input due to the non-ideal nature of the communication channel.

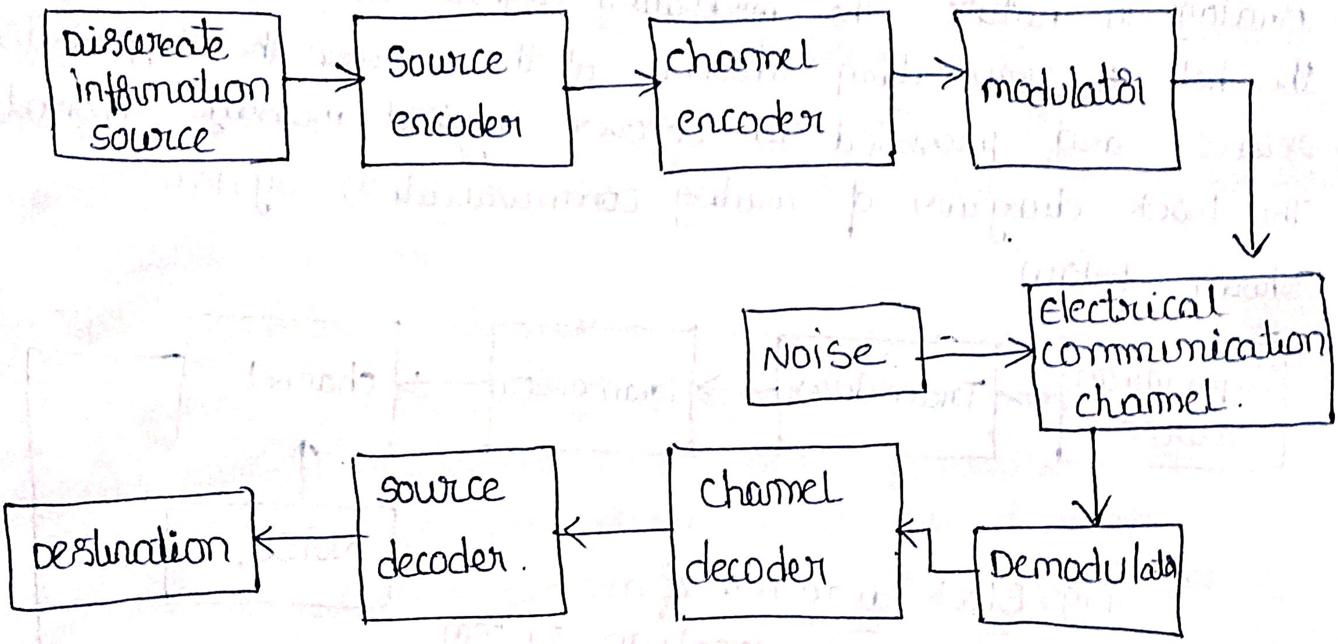


Fig. Elements of digital communication system.

In addition to this, the information bearing signals is also corrupted by unpredictable electrical signals from both manmade and natural causes. Thus, the smearing and the noise introduce errors in the information being transmitted and limits the rate at which information can be communicated from the source to the destination.

Discrete information source :-

Information source may be classified into two categories, based upon the nature of their output. i.e analog information sources and discrete information sources. In case of analog communication, the information source is analog.

In case of digital communication, the information source produces a message signal which is not continuously varying with time. The output of discrete information sources such as a teletype or the numerical output of a computer consists of a sequence of discrete symbols or letters. Discrete information sources are characterized by the following parameters,

i) **Source alphabet** :- These are the letters, digits or special characters available from the information source.

ii) **Symbol rate** :- It is the rate at which the information source generates source alphabets. It is generally represented in symbols / sec (unit).

iii) **Source alphabet probabilities** :- Each source alphabet from the source has independent occurrence rate in the sequence. An as example, letters A, E, I.. occur frequently in the sequence. Hence probability of the occurrence of each source alphabet can become one of the important property which is useful in digital communication.

iv) **Probabilistic dependence of symbols in a sequence** :-

The information carrying capacity of each source alphabet is different in a particular sequence. This parameter defines average information content of the symbols. This means that the source information rate is the product of symbol rate and source entropy

Information rate = Symbol rate x Source Entropy
(Bits|sec). (Symbols|sec) (Bits|symbol)

Thus, the information rate represents minimum average data rate required to transmit information from source to the destination.

Source Encoder and decoder :-

The symbols produced by the information source are given to the source encoder. These symbols cannot be transmitted directly. They are first converted into digital form by the source encoder. Each binary '1' and '0' is known as a bit. The group of bits is called a codeword. The source encoder assigns codewords to the symbols. Source Encoders must have following important parameters.

i) Block Size :-

Block size describes the maximum number of distinct codewords which can be represented by a source encoder. This depends on the number of bits in the codeword. As an example, the block size of 8-bits source encoder will be 2^8 i.e 256 code words.

ii) **Average data rate** :- Average data rate means the number of bits per second generated by a source encoder.

Data rate = Symbol rate x Codeword length.

iii) Codeword length :-

Code word length is the no. of bits used to represent each codeword. As an example, if 8 bits are assigned to each codeword.

iv) Efficiency of the encoder :-

The efficiency of the encoder is the ratio of minimum source information rate to the actual output data rate of the source encoder.

Channel Encoder and decoder :-

After converting the message signal in the form of binary sequence by the source encoder, the signal is transmitted through the channel. The communication channel adds noise and interface to the signal being transmitted. Hence errors are introduced in the binary sequence received at the receiver end.

Channel encoder is mainly used to avoid the errors in digital communication system is done by adding redundant bits. channel encoder is characterized by certain parameters.

1) The coding rate that depends upon the redundant bits added by the channel encoder.

2) The coding method used.

3) coding efficiency which is the ratio of data rate at the input to the data rate at the output of the Encoder.

4) Error control capabilities.

5) Feasibility of the encoder and decoder.

This means that the channel encoder and decoder serve to increase the reliability of a received signal.

Digital modulators and demodulators :-

Digital continuous wave modulators are ASK, FSK, & PSK. These modulators use a continuous carrier wave, therefore they are also known as digital cw modulators. At the receiver end. the digital demodulator converts the input modulated signal into the sequence of binary bits.

A digital modulation method must have following important parameters.

i) Bandwidth needed to transmit the signal.
2) probability of symbol or bit error.
3) Synchronous or Asynchronous method of detection.
4) complexity of implementation.

communications channel :-

The connection between transmitter and receiver is established through a communication channel. The communication can take place through wirelines, wireless or fiber optic channel. Each and every communication channel has some inherent problems. These are

i) signal Attenuation :-

The signal attenuation in channel occurs due to the internal resistance of the channel and fading of the signal.

ii) Amplitude and phase distortion :-

The transmitted signal is distorted in Amplitude and phase due to the non-linear characteristics of the communication channel.

iii) Additive noise interference:-

It is produced due to internal solid state devices and resistors etc. used to implement a communication system.

iv) multipath distortion :- It occurs mostly in wireless communication channels.

Destination :- It is the final state where the information is reached.

Advantages of digital communication system :-

→ These communication systems are simpler and cheaper.

→ only permitted receivers can receives data.

→ It having a wide dynamic range.

→ Noise can be easily tolerated.

→ In this communication, channel coding is used therfore the errors may be detected and corrected at the receivers.

Disadvantages :-

→ Transmission bandwidth is required for digital communication.

→ Digital communication needs synchronization in case of synchronous modulation.

Comparison of Analog and digital modulation :-

Analog modulation.	Digital modulation.
1. Transmitted modulated signal is analog in nature.	1. Transmitted signal is digital (train of digital pulses).
2. Amplitude, frequency or phase variations in the transmitted signal represent the information or message.	2. Amplitude, width or position of the transmitted pulses is constant. The message is transmitted in the form of code words.
3. Noise immunity is poor for Am, but improved for FM and PM.	3. Noise immunity is Excellent.
4. It is not possible to separate out noise and signal. Therefore repeaters cannot be used.	4. It is possible to separate signal from noise. Therefore, repeaters can be used.
5. coding is not possible.	5. coding techniques can be used to detect & correct the errors.
6. Bandwidth required is lower than that for the digital modulation methods.	6. Due to higher bit rates, higher channel bandwidth is required.
7. FDM is used for multiplexing.	7. TDM is used for multiplexing.
8. Not suitable for transmission of secret information in military applications.	8. Due to coding techniques, it is suitable for military applications.
9. Analog modulation systems are AM, FM, PM, PAM & PWM etc.	9. Digital modulation systems are PCM, DM, ADM, DPCM. etc.

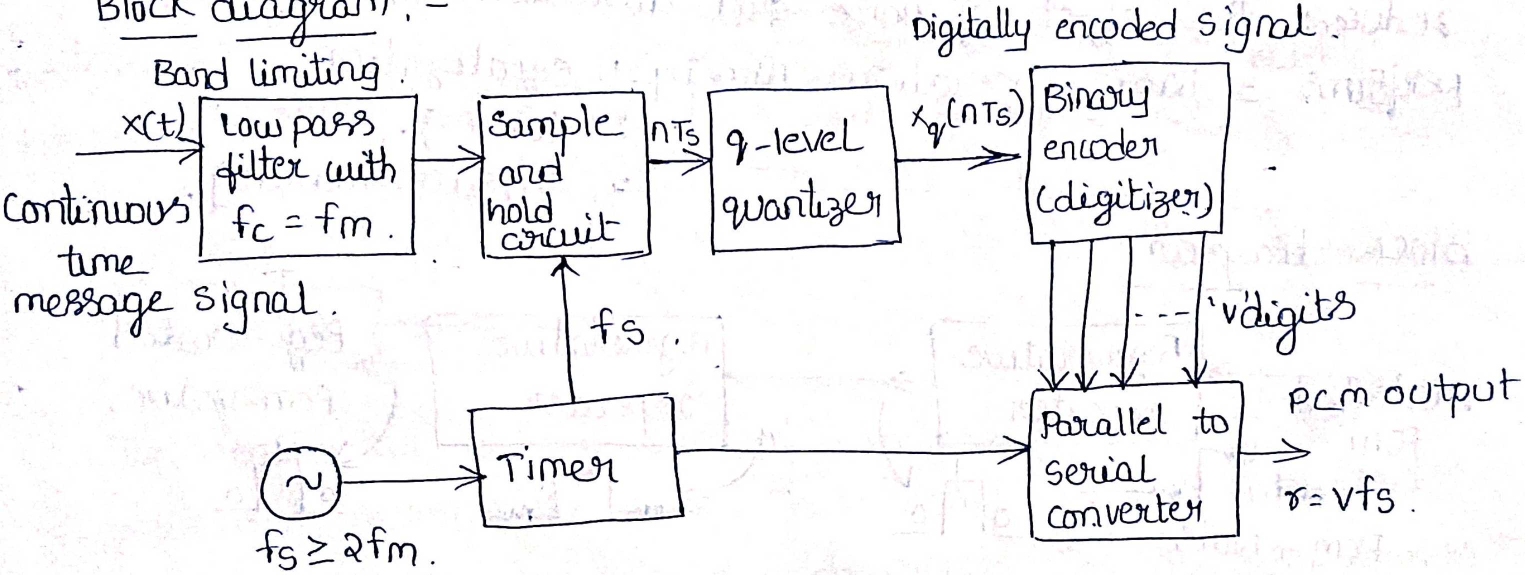
Pulse code Modulation :- (PCM)

Pulse - code modulation is known as a digital pulse modulation technique. The PCM output is in the form of digital pulses of constant amplitude, width and position. The PCM system mainly consists of three main parts.

- * PCM Transmitter
- * Transmission path
- * Receiver.

* PCM Transmitter :-

Block diagram :-



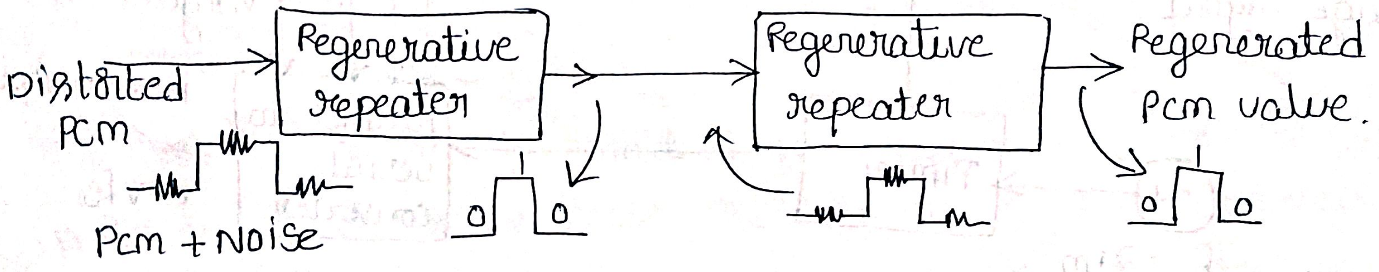
In PCM transmitter the essential operations ~~are~~ ~~in~~ are PCM sampling, quantizing and encoding.

Here the signal x(t) is first passed through the low-pass filter of cut-off frequency fm. This low pass filter blocks all the frequency components above the fm. The sample and hold circuit samples this signal at the rate of fs. The output of sample and hold circuit is denoted by x(nTs). This x(nTs) is a discrete signal. Next, the quantizer compares input x(nTs) with its fixed digital levels and assigns any one of the digital level to $x(nT_s)$ with its fixed digital levels. Now, the quantized signal level $x_q(nT_s)$ is given to binary encoder. This encoder converts input signal to v digits binary word. parallel to serial converter converts parallel form of data to serial form

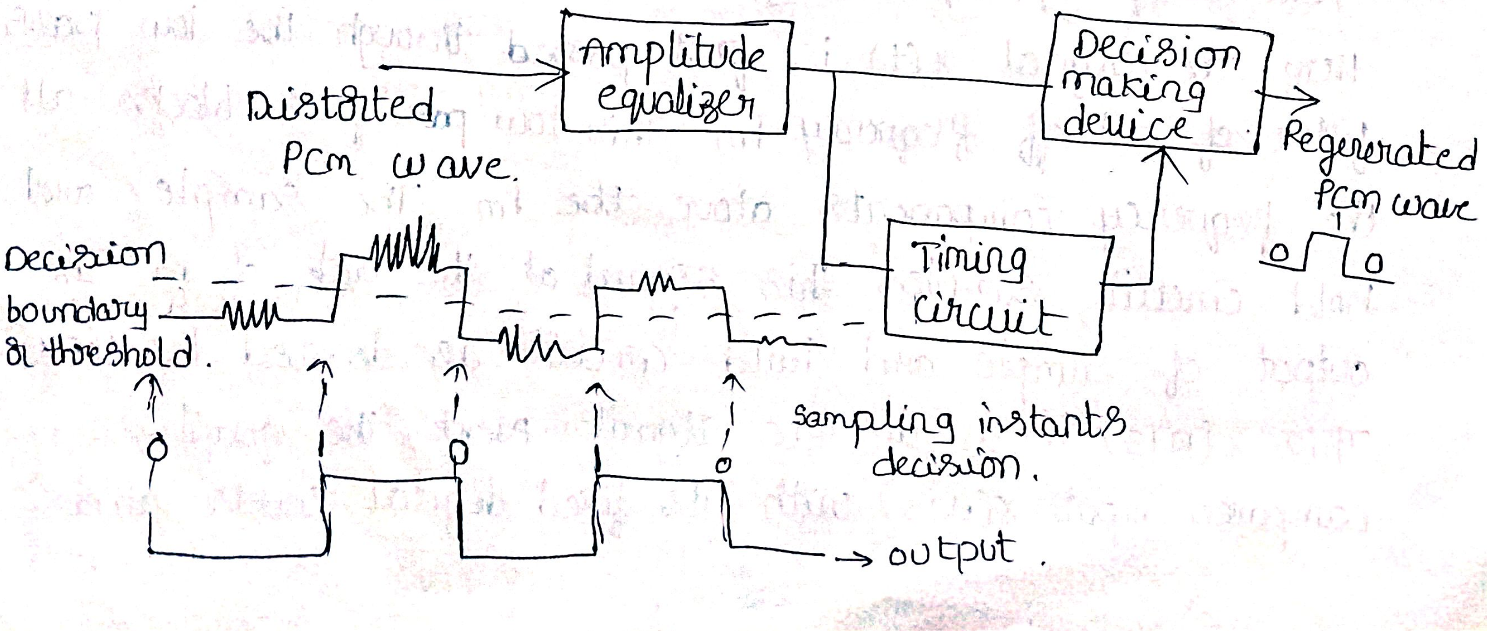
* Transmission path :-

The path between the PCM Transmitter and PCM receiver over which PCM signal travel is called as PCM Transmission path. Here we use regenerative repeaters which reduces the effect of noise and distortion. The regenerative performs 3 Basic operations namely.
1. equalization
2. timing.
3. decision making.

Block diagram :-



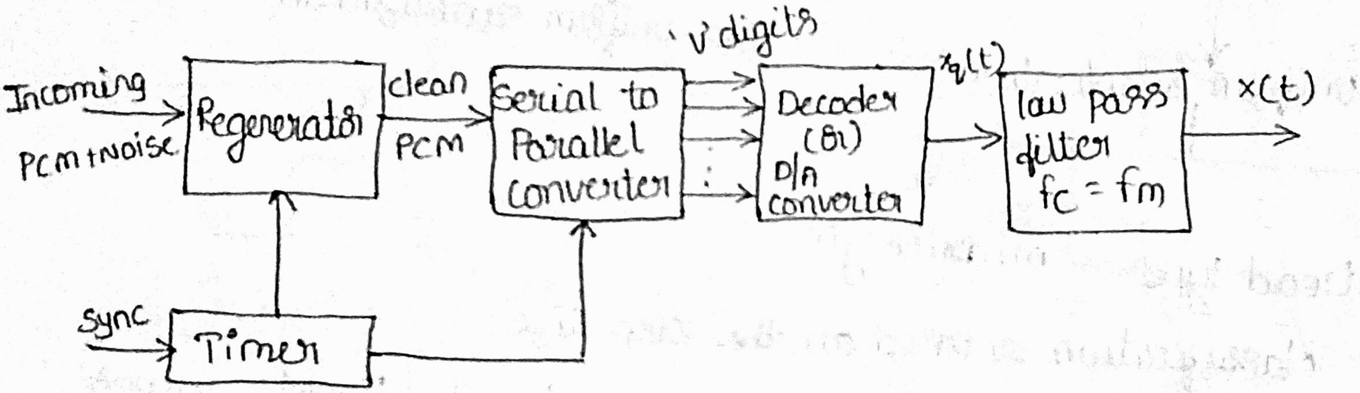
Block diagram of regenerative repeater :-



The amplitude equalizer shapes the distorted PCM wave so as to compensate for the effects of amplitude and phase distortions. The Timing circuit produces periodic pulse train. The decision making uses this pulse train for sampling the equalized PCM pulses The decision is made by comparing equalized PCM with reference level called decision threshold.

* PCM Receiver :-

Block diagram :-

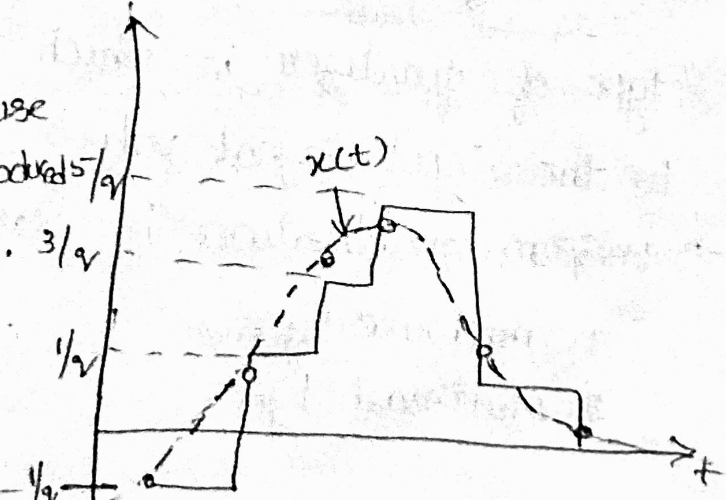


The regenerator at the start of PCM receiver reshapes the pulse and removes the noise. This signal is then converted to parallel digital words of each sample. The decoding process generating an analog signal it will denoted by $x_q(t)$. is allowed to pass through a low pass reconstruction filter to get the appropriate original message signal denoted as $x(t)$..

* It is impossible to reconstruct exact original signal x(t) because permanent quantization error introduced during quantization at the Tx.

'E' can be reduced by increasing binary levels.

$\boxed{q\uparrow \quad v\uparrow \quad B.W\uparrow}$

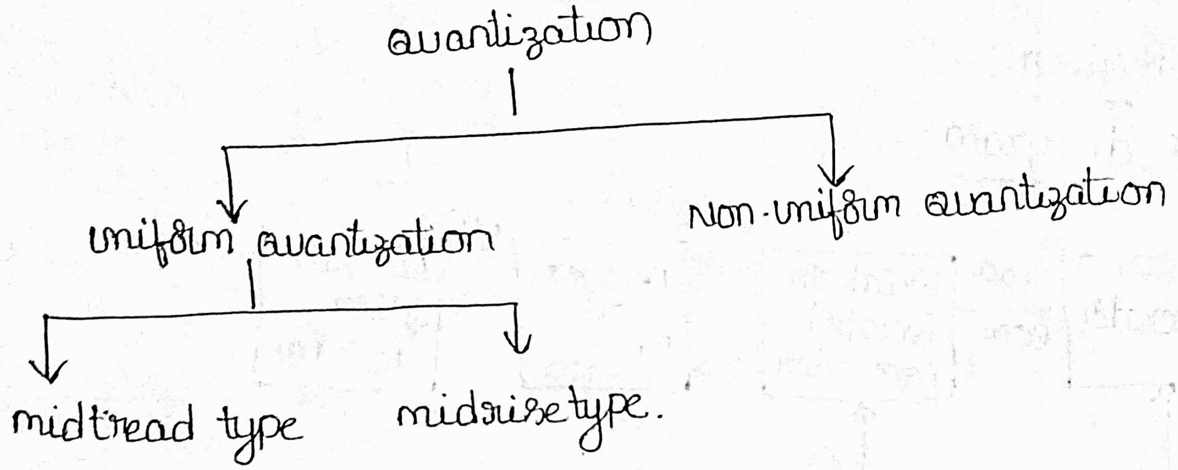


Reconstructed waveform

Quantizer :-

A quantizer compares discrete time input x(nTs) with its fixed digital levels. It assigns any one of the digital level to x(nTs) with its fixed digetal levels which results in minimum error. This error is called quantization error.

Classification of quantization process :-



This classification is based on the step size.

Step size :- The difference between two adjacent discrete values is called a "quantum" or stepsize

i) **uniform quantizer :-** A uniform quantizer is that type of quantizer in which the 'step size' remains same throughout the input range.

ii) non **uniform quantizer :-** A non uniform quantizer is that type of quantizer in which the 'step-size' varies according to the input signal values.

→ uniform quantization is classified as.

1. mid rise type
2. midtread type

- * In the input and output characteristic of midtread type the origin lies in the middle of a tread of stair case like graph.
- * In the input and output characteristic of midrise type the origin lies in the middle of rising part of stair case like graph.

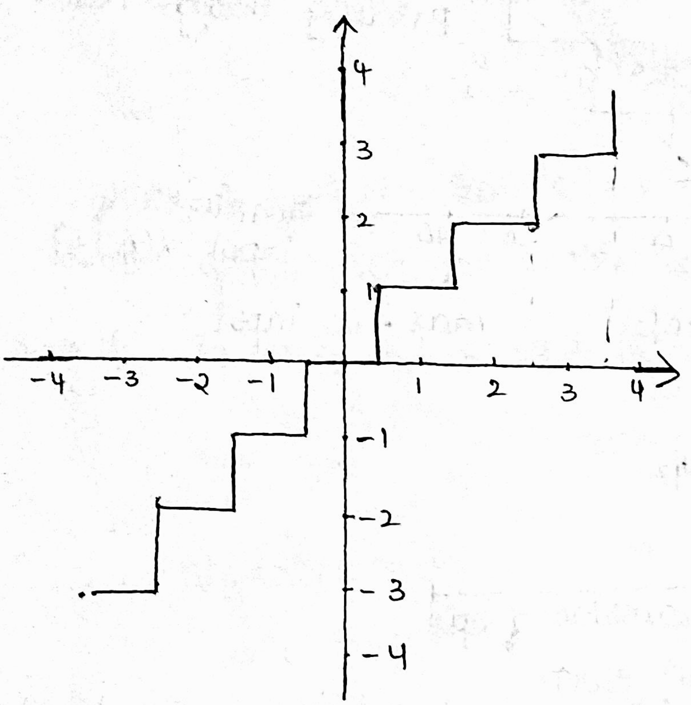


fig (a) midtread

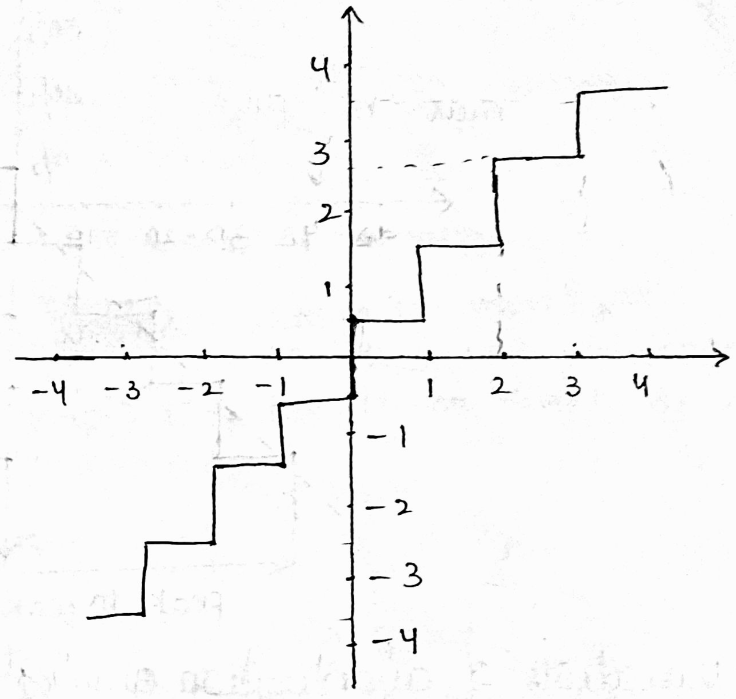


fig (b) midrise.

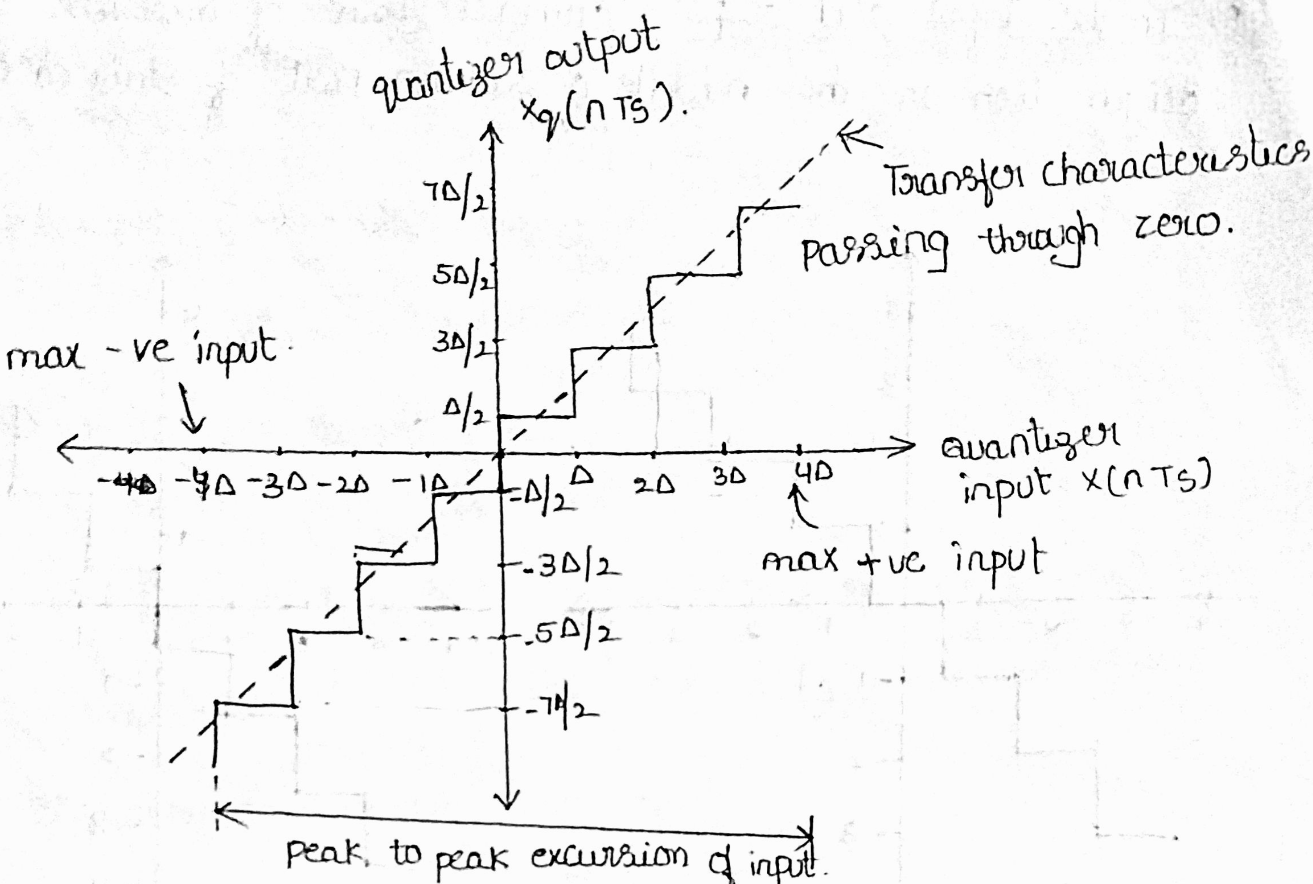
→ **working principle of quantizer :-**

For this purpose, we shall consider uniform quantizer of midrise type. The transfer characteristics of a uniform quantizer of midrise type. let us assume that the input to the quantizer $x(nT_s)$ varies from 4Δ to $+4\Delta$. This means that the peak to peak value of $x(nT_s)$ will be between -4Δ to $+4\Delta$. Here Δ is the step size. The fixed digital levels are avaliable at $\pm \Delta/2$, $\pm 3/2\,\Delta$, $\pm \frac{5}{2}\Delta$, $\pm \frac{7}{2}\Delta$. These levels are avaliable at quantizer because of its characteristics.

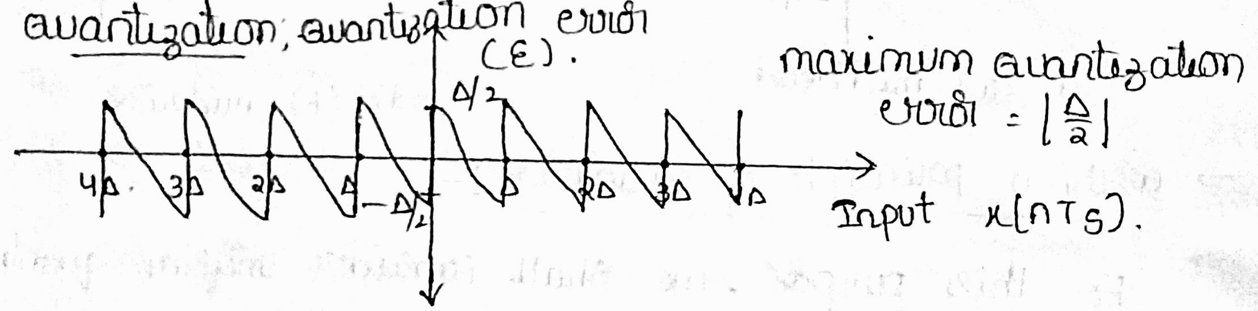
If $x(nT_s) = 4\Delta$ then $x_q(nT_s) = \frac{7}{2}\Delta$

$x(nT_s) = -4\Delta$ then $x_q(nT_s) = -\frac{7}{2}\Delta$

Transfer characteristic



Variation of quantization, quantization error



quantization error is defined as the error that arise when digital levels are assigned to the input levels.

quantization error $\varepsilon = x_q(nTs) - x(nTs)$.

In above figure, it may ~~be~~ also be observed that.

for $\Delta < x(nTs) < 2\Delta$, $x_q(nTs) = \frac{3}{2}\Delta$

$-\Delta < x(nTs) < -2\Delta$, $x_q(nTs) = \frac{-3}{2}\Delta$.

This means that the maximum quantization error will be $\pm \frac{\Delta}{2}$

$$\boxed{\varepsilon_{max} = \left|\frac{\Delta}{2}\right|}$$

→ Transmission bandwidth in PCM system :-

In PCM system the quantizer use 'v' number of binary digits to represent each level. Then, the number of levels that may be represented by 'v' digits will be $q = 2^v$.

For example, if v=4 bits, the total number of levels will be

$$q = 2^v = 2^4 = 16 \text{ levels}.$$

Number of bits per sample = v

number of samples per second = fs.

No. of bits per second := no. of bits per samples x. number of samples per second.

= 'V' bits per sample x f_s samples per second

∴ signaling rate in PCM, $r = vf_s$.

$$f_s \geq 2f_m.$$

The bandwidth required for PCM system must be greater than half of signalling rate

$$BW \geq \frac{1}{2} r$$

$$BW \geq \frac{1}{2} vf_s \Rightarrow B.W \geq \frac{1}{\cancel{2}} v.\cancel{2} f_m$$

$$\therefore BW \geq vf_m.$$

→ Quantization Noise / Error in PCM :-

Because of quantization, inherent errors are introduced in the signal. This error is called quantization error. The quantization error is given as

$$E = x_q(nTs) - x(nTs).$$

let us consider amplitude ranges from $-X_{max}$ to $+X_{max}$. in uniform quantization. Then Total amplitude range is.

$$= X_{max} - (-X_{max})$$

$$= 2X_{max}.$$

If total amplitude range is divided into 'q' levels. then step size Δ is

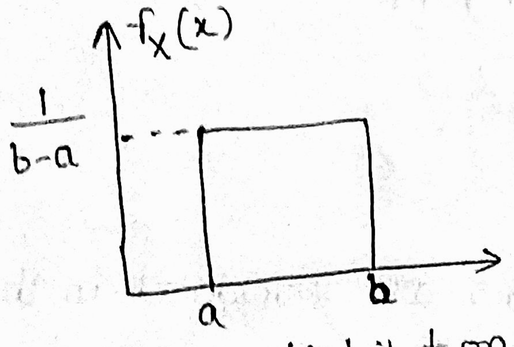
$$\Delta = \frac{2.X_{max}}{q} \qquad (\because X_{max} = 1, \; -X_{max} = -1)$$

Therefore, step size would be $\Delta = \frac{2}{q}$.

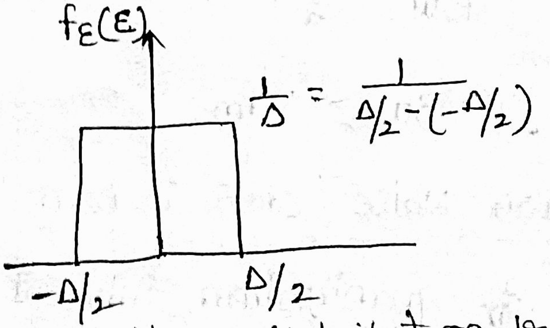
Now, if step size 'Δ' is considered as sufficiently small, then it may be assumed that the quantization error 'ε' will be an uniformly distributed random variable. The maximum quantization error is

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right| \qquad -\frac{\Delta}{2} \leq \varepsilon_{max} \leq \frac{\Delta}{2}.$$

Hence, over the interval $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$ quantization error may be assumed as an uniformly distributed random variable.



a uniform distribution



A uniform distribution for quantization error.

An uniformly distributed random variable 'x' over an interval (a, b) the probability density function of uniformaly distributed random variable 'x' is given as.

$$f_X(x) = \begin{cases} 0 & \text{for } X \le a \\ \frac{1}{b-a} & \text{for } a < x \le b \\ 0 & \text{for } X > b \end{cases}$$

Thus the help of equation, the probability density function (PDF) for quantization error ε may be defined as.

$$f_\varepsilon(\varepsilon) = \begin{cases} 0 & \text{for } \varepsilon \le \frac{\Delta}{2} \\ \frac{\Delta}{2} & \text{for } -\frac{\Delta}{2} < \varepsilon < \frac{\Delta}{2} \\ 0 & \text{for } \varepsilon > \frac{\Delta}{2} \end{cases}$$

The noise power is expressed as

$$\text{noise power} = \frac{V^2_{noise}}{R}.$$

V^2_{noise} is mean square value of noise (ε).

we know that
$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

similarly.

$$E[\varepsilon^2] = \int_{-\infty}^{\infty} \varepsilon^2 f_\varepsilon(\varepsilon)\, d\varepsilon$$

$$= \int_{-\Delta/2}^{\Delta/2} \varepsilon^2 \cdot \frac{1}{\Delta}\, d\varepsilon = \frac{1}{\Delta}\left[\frac{\varepsilon^3}{3}\right]_{-\Delta/2}^{\Delta/2}$$

$$= \frac{1}{3\cancel{\Delta}} \times \left[\frac{\Delta^{\cancel{3}}}{4}\right] = \frac{\Delta^2}{12}.$$

$\therefore$ Quantization noise or error $= \frac{\Delta^2}{12}$

Signal to noise ratio of Pcm system :-

$$\frac{S}{N} = \frac{\text{signal power (Normalized)}}{\text{Noise power (Normalized}} = \frac{P}{\Delta^2/12}$$

quantization Noise $= \frac{\Delta^2}{12} \quad \left(\Delta = \frac{2X_{max}}{2^V} \right)$

$$\Delta^2 = \left(\frac{2x_{max}}{2^V} \right)^2 = \frac{4X_{max}^2}{2^{2V}} \qquad (X_{max} = 1)$$

$$\text{quantization} = \frac{\cancel{4}}{2^{2v}} \cdot \frac{1}{\cancel{12}_3} = \frac{1}{3.}$$

Then quantization Noise is $= \frac{\cancel{4} X_{max}^2}{2^{2V}} \cdot \frac{1}{12}$

Then $\frac{S}{N} = \frac{P}{\frac{4X_{max}^2}{2^{2V}} \cdot \frac{1}{\cancel{12}\,3}} = \frac{3P}{X_{max}^2} \cdot 2^{2V}$

$$\frac{S}{N} \leq 3 \times 2^{2V}. \qquad (X_{max} = 1)$$

$$(P \leq 1)$$

The signal to noise ratio in db.

$$\left(\frac{S}{N} \right)_{dB} \leq 10 \log \left(3 \times 2^{2V} \right)$$

$$\leq 10 \log_{10} 3 + 20 V \log_{10} 2 .$$

$$\left(\frac{S}{N} \right)_{dB} \leq 4.8 + 6V.$$

Signal to noise ratio for sinusoidal input

we Know $\frac{S}{N} = 3P2^{2V}$

consider modulating signal is a sinusoidal voltage having peak amplitude X_{max}.

$$P = \frac{V^2}{R} \quad \text{let } (R = 1)$$

$$V_{rms} = \frac{A_m}{\sqrt{2}} = \frac{X_{max}}{\sqrt{2}}$$

$$P = \frac{X_{max}^2}{2} = \frac{1}{2} \quad (\because X_{max} = 1)$$

$$\left(\frac{S}{N}\right)_{dB} = 10 \log \left(3 \times P \times 2^{2V}\right)$$

$$= 10 \log \left(\frac{3}{2} \times 2^{2V}\right)$$

$$= 1.8 + 6V.$$

Non uniform quantization :-

If the quantizer characteristics is non-linear and the step size is not constant in stead if it is variable, dependent on the amplitude of input signal then the quantization is known as non-uniform quantization. In non-uniform quantization, the step size is reduced with the reduction in signal level. for weak signals (P<<1), the step size is small, therefore the quantization noise reduces, to improve the signal to quantization noise ratio for weak signals. The step size is Thus varied according to the signal level to keep the signal to noise ratio adequately high. This is non-uniform quantization. The non-uniform quantization is practically achieved through a process called "companding".

→ Companding :-

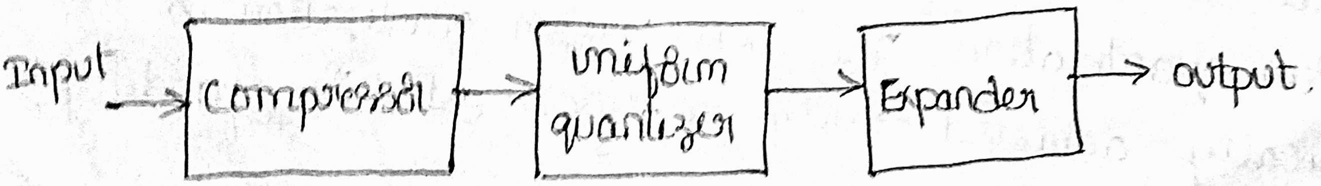
It is used to improve the signal to quantization noise ratio of weak signals.

$$N = \frac{\Delta^2}{12}$$

In uniform quantization, once the step size is fixed, the quantization noise power remains constant and signal power is not constant. It is proportional to the square of signal amplitude. Hence signal power will be small for weak signals, but quantization noise power is constant. Therefore, the signal to quantization noise for the weak signals is very poor. Companding is a term derived from two words.

Companding = Compressing + Expanding.

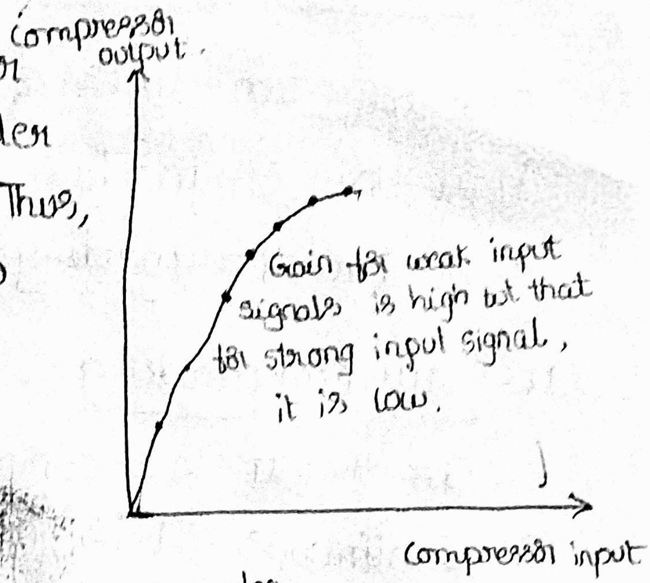
In practice, it is difficult to implement the non-uniform quantization because it is not known in advance about the changes in the signal levels. Therefore, a particular method is used. Therefore the weak signals are amplified and strong signals are attenuated. This process is called compression and the block is called compressor.



At the receiver we use expansion. The circuit used for providing expansion is called as an Expander. The compression of signal at the transmitter and expansion at the receive is called companding.

Compressor characteristic :-

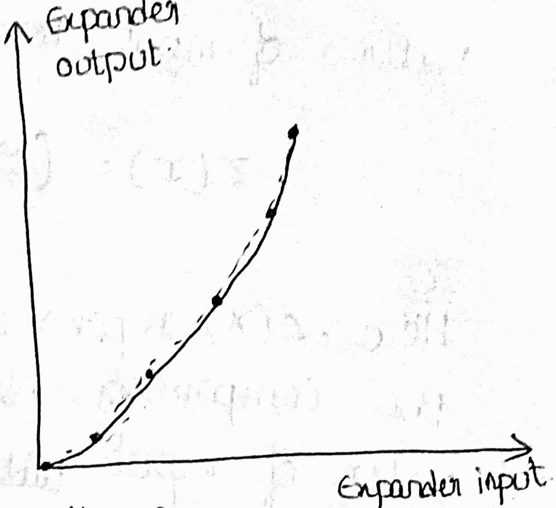
The compressor provides a higher gain to the weak signals and smaller gain to the strong input signals. Thus, weak signals are artifically boosted to improve the signal to quantization noise ratio.



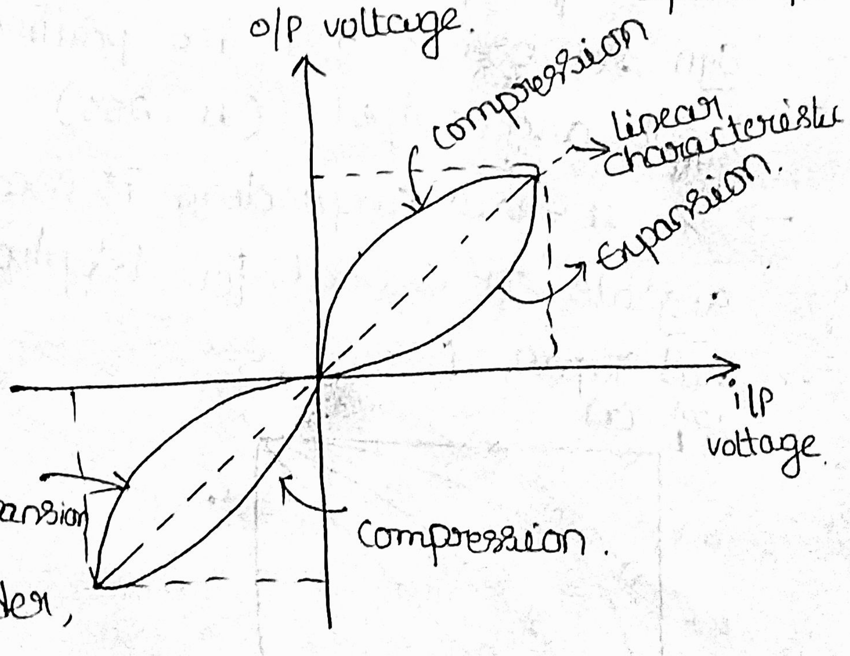
Gain for weak input signals is high but that for strong input signal, it is low.

Expander characteristics :-

This is exactly inverse of compressor characteristics. The artificially boosted signals by the compressor are brought back to their original amplitudes at the receiver end.



Compander characteristics :-



It is the combination of both compressor and Expander characteristics. Due to inverse nature of compressor and Expander, the overall characteristics of the compander is a straight line.

Different types of compressor :-

There are two types of companding in Pcm system.

1) μ - law companding.
2) A - Law companding.

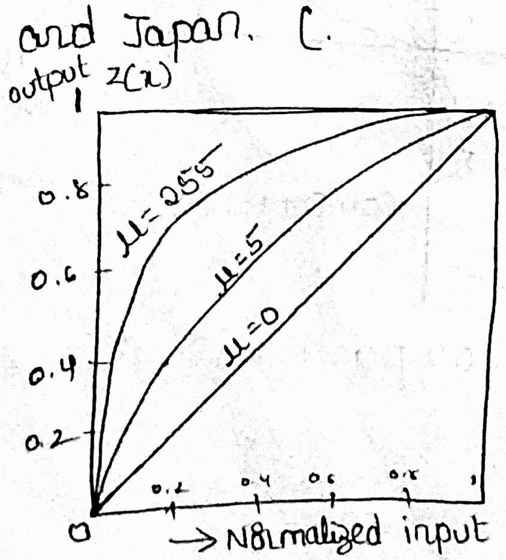
μ - law companding :-

In the μ-law companding, the compressor characteristic is continuous. It is approximately linear for smaller values of input levels and logarithmic for high input levels.

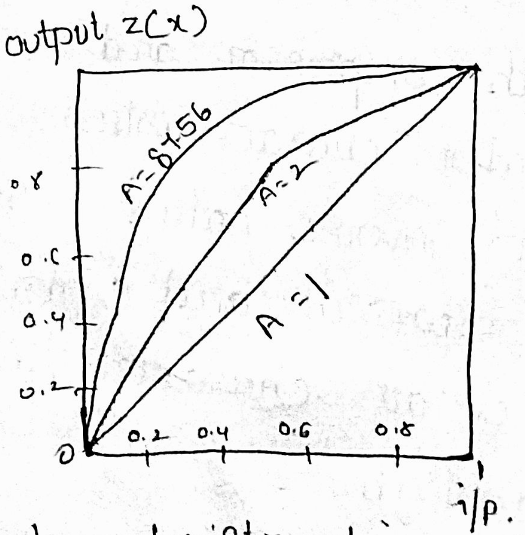
$$Z(x) = (\text{Sgn } x) \frac{\ln(1+\mu|x|/x_{max})}{\ln(1+\mu)} \quad 0 < |x|/x_{max} \leq 1.$$

Here, $Z(x)$ represents the output and x is the input to the Compressor. Also $|x|/x_{max}$ represents the normalized value of input with respect to the maximum value X_{max}. 'Sgn' represents ± 1. i.e positive and negative values of input and output. ($\mu = 255$).

→ The μ - law companding is used for speech and music signals. It is used for telephone systems in US, Canada and Japan. C.



characteristic of a μ - law compressor.



characteristic of a A - law compressor.

A-law companding :-

In the A-law companding, the compressor characteristic is piecewise, made up of a linear segment for low levels inputs and a logarithmic segment for high level inputs. The A-law companding is used for PCM telephone system in europe. practically $A = 87.56$.

$$\frac{Z(x)}{X_{max}} = \begin{cases} \dfrac{A|x|/x_{max}}{1+\log_e A} & \text{for } 0 \le \dfrac{|x|}{x_{max}} \le 1 \\ \dfrac{1+\log_e[A|x|/x_{max}]}{1+\log_e A} & \text{for } \dfrac{1}{A} \le \dfrac{|x|}{x_{max}} \le 1. \end{cases}$$

Applications of PCM :-

1. PCM is used in telephony.
2. PCM is used in space communication.

Advantages :-

1. PCM Provides high noise immunity.
2. we can store the PCM signal due to its digital nature.
3. we can use various coding techniques so that only the desired person can decode the received signal.
4. Due to digital nature of the signal, we can place repeaters between the transmitter and the receivers. Repeaters further reduces the effect of noise.

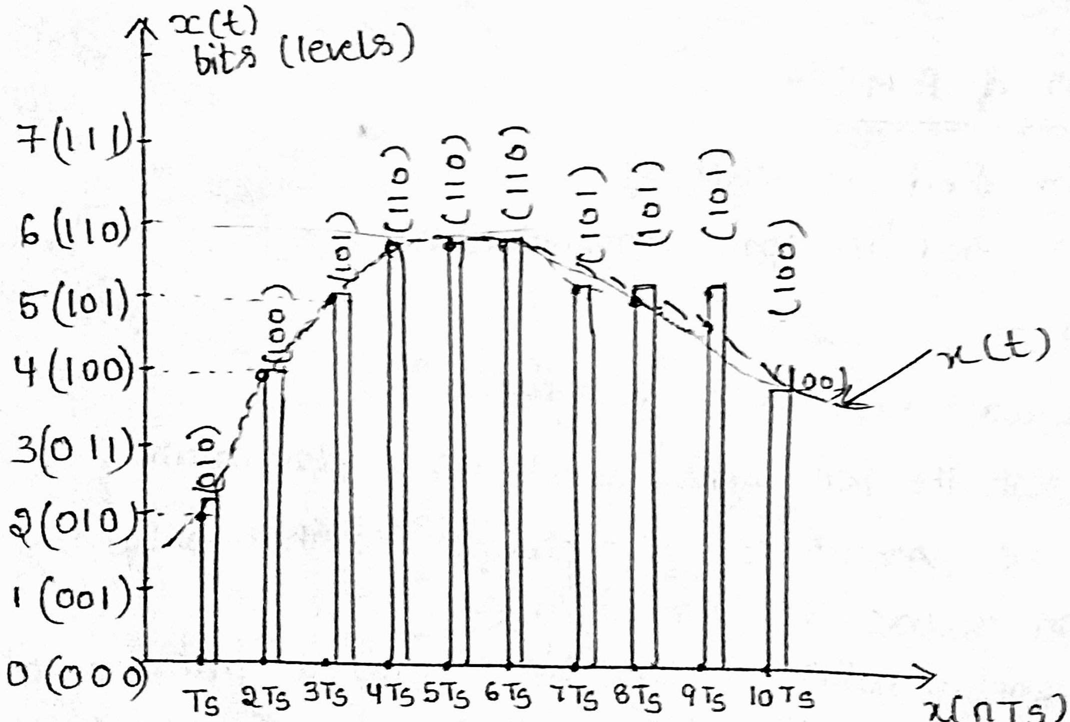
Drawbacks of PCM :-

1. The encoding, decoding and quantizing circuitry of PCM is complex.
2. PCM requires a large bandwidth as compared to the other systems.

· Differential pulse code modulation (DPCM):-

i) **Reason to use DPCM:-** The samples of a signals are highly corelated with each other. This is due to the fact any signal does not change fast. This means that its value from present sample to next sample does not differ by large amount. The adjacent samples of the signal carry the same information with a little difference. when these the samples are encoded by a standard PCM system, the resulting encoded signal contains some redundant information.

ii) **Redundant information in PCM :-**

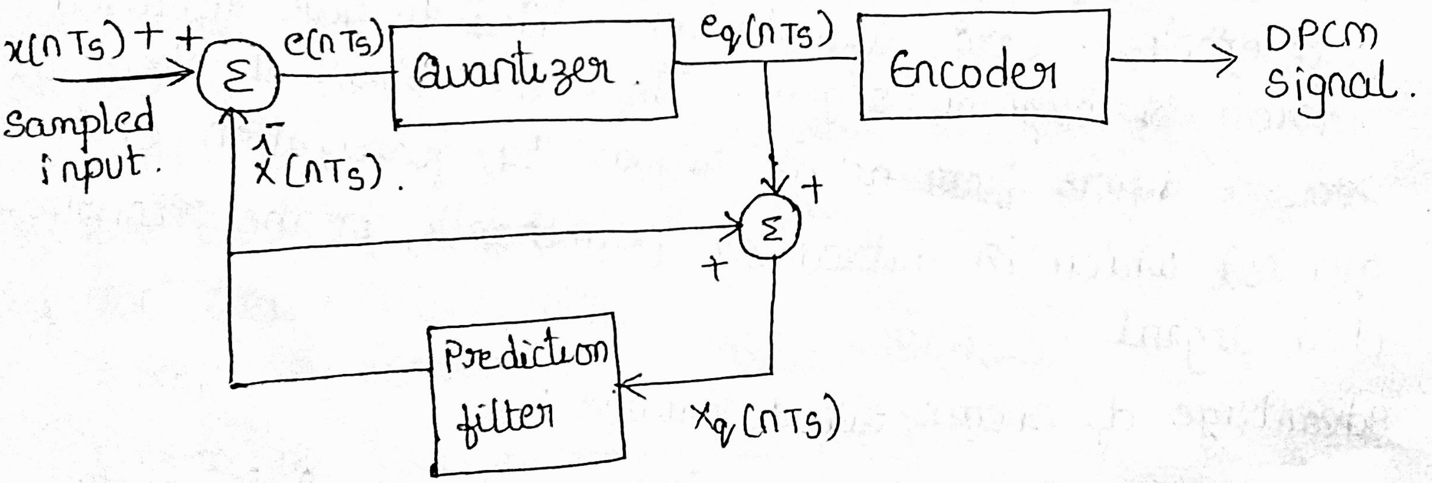


The signal is sampled by flat top sampling at intervals Ts, 2Ts, 3Ts ... nTs. If we observe, 4Ts, 5Ts and 6Ts are encoded to same value of (110). This information can be carried by one sample. But three samples are carrying the information. This is redundant. If this redundancy is reduced, then over all bitrate will decrease and no. of bits to be transmitted in one sample will also reduce.

If the redundancy is reduced, then overall bit rate will decrease and number of bits required to transmit one sample will also be reduced. This type of digital pulse modulation scheme is known as Differential pulse code modulation (DPCM).

DPCM transmitter :-

It works on the principle of prediction. The value of present sample is predecated from past sample. The sampled signal is denoted by $x(nTs)$ and predicated signal by $\hat{x}(nTs)$. The predicted value is produced by prediction filter. The quantizer output is added with previous prediction and given as input to prediction filter $x_q(nTs)$. we can observe that $e_q(nTs)$ is small and can be encoded by using small no.of bits. Thus bits per sample is reduced.



prediction error is

$$e(nTs) = x(nTs) - \hat{x}(nTs).$$

quantizer output

$$e_q(nTs) = e(nTs) + q(nTs)$$

$q(nTs)$ is the quantization error.

input to ~~quan~~ prediction filter.

$$x_q(nTs) = \hat{x}(nTs) + e_q(nTs).$$

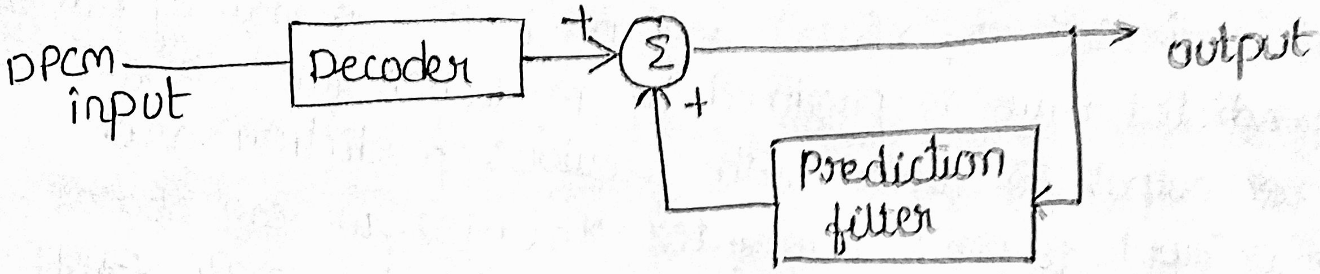
$$= \hat{x}(nTs) + e(nTs) + q(nTs)$$

$$= \hat{x}(nT_s) + x(nT_s) - \hat{x}(nT_s) + q(nT_s)$$

$$= x(nT_s) + q(nT_s)$$

Hence the prediction filter input does not depends on prediction filter characteristics.

DPCM Receiver :-



The decoder first reconstruct the quantized error signal from incoming binary signal The prediction filter o/p and quantized error signals are added to give quantized version of original signal. Thus the signal at the receiver differs from actual signal by quantization error $q(nT_s)$ which is introduced permenantly in the reconstructed signal.

Advantage of DPCM : salient features :-

1. As the difference between $x(nT_s)$ and $\hat{x}(nT_s)$ is being encoded and transmitted by the DPCM technique, A small difference voltage is to be quantized and encoded.
2. This will requires less number of quantization levels and hence less number of bits to represent them.
3. Thus signaling rate and bandwidth of a DPCM system will be lessthan that of PCM.

Delta modulation :-

i) Reason to use Delta modulation :-

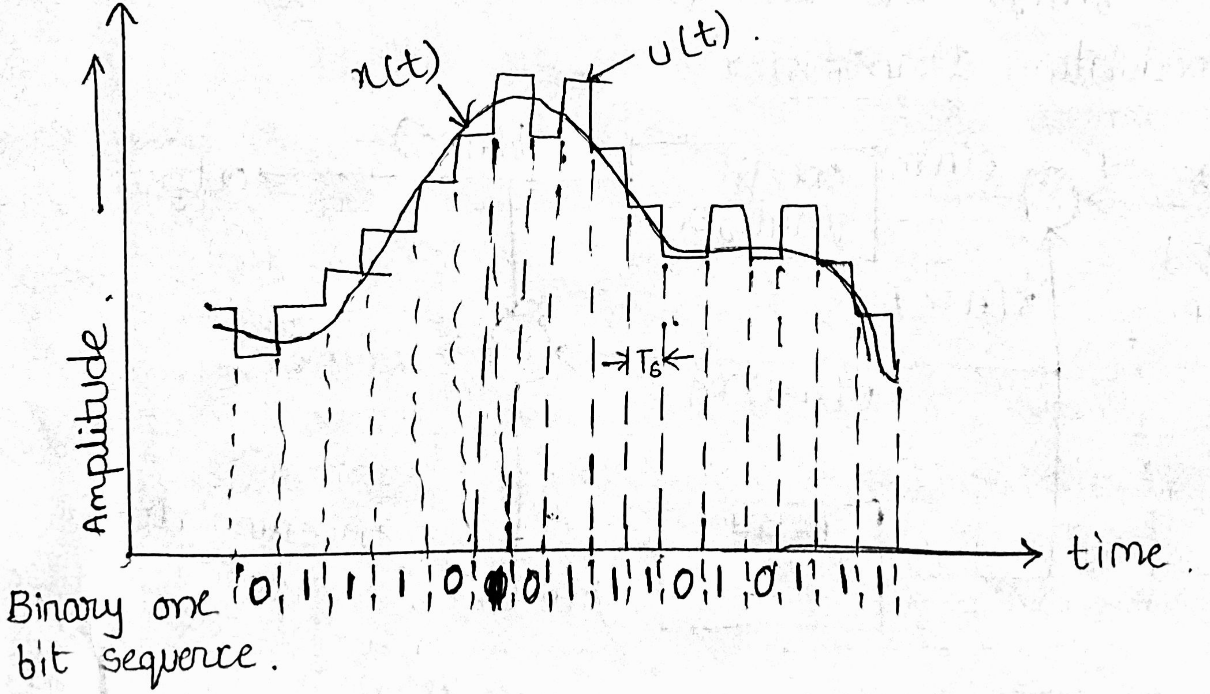
we have observed in PCM that it transmits all the bits which are used to code a sample. Hence, signaling rate and transmission channel bandwidth are quite large in PCM. To overcome this problem. Delta modulation is used.

ii) working principle :-

Delta modulation transmits only one bit per sample. Hence the present sample value is compared with the previous sample value and this result whether the amplitude is increased or decreased is transmitted. Input signal x(t) is approximated to step signal by the delta modulator. This step size is kept fixed. The difference between input signal x(t) and staircase approximated signal is confined to two levels i.e +Δ and -Δ. If difference is positive, then signal is increased by one step. If difference is negative, then signal is reduced by one step.

If step is increased, '1' is transmitted.

If step is decreased, '0' is transmitted.



Mathematical Expressions :-

Thus, the principle of delta modulation can be explained with the help of few equations as shown.

The error is given as,

$$e(nT_S) = x(nT_S) - \hat{x}(nT_S).$$

where $e(nT_S)$ = error at present sample

$x(nT_S)$ = sampled signal of $x(t)$

$\hat{x}(nT_S)$ = last sample approximation of the staircase waveform.

If we assume $u(nT_S)$ as the present sample approximation of staircase output, then.

$$u[(n-1)T_S] = \hat{x}(nT_S).$$

Now, let us define a quantity $b(nT_S)$ in such a way that

$$b(nT_S) = \Delta \ \mathrm{sgn}\,[e(nT_S)].$$ T_S = Sampling interval.

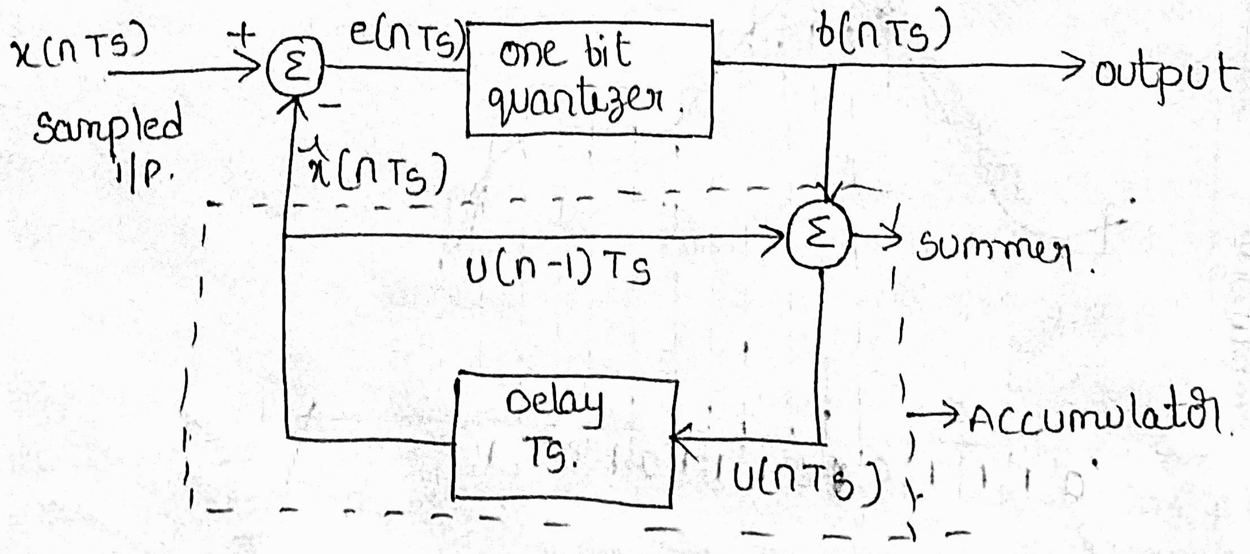
In other words.

$$b(nT_S) = \begin{cases} +\Delta & \text{if } x(nT_S) \geq \hat{x}(nT_S) \\ -\Delta & \text{if } x(nT_S) < \hat{x}(nT_S). \end{cases}$$

Also, if $b(nT_S) = +\Delta$ then '1' is transmitted and

$b(nT_S) = -\Delta$ then '0' is transmitted.

Delta modulation transmitter :-



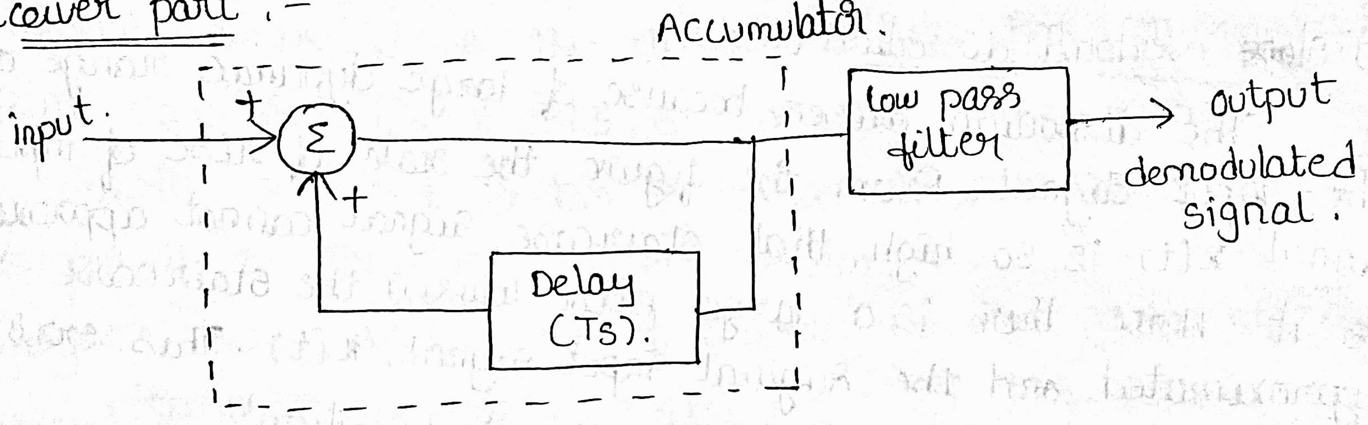
The summer in accumulator adds quantizer o/p with previous sample approximation. This gives the present sample approximation i.e.

$$u(nT_S) = u[(n-1)T_S] + [\pm\Delta]$$

(or) $u(nT_S) = u[(n-1)T_S] + b(nT_S)$.

Thus, depending on the sign of $e(nT_S)$. one bit quantizer generates an output of $+\Delta$ or $-\Delta$. If stepsize is $+\Delta$, then binary '1' is transmitted. If stepsize is $-\Delta$ then binary '0' is transmitted.

Receiver part :-



At receiver end, the accumulator and LPF are used. The accumulator generates the stair case approximated signal output and is delayed by one sampling period Ts. It is added to the input signal. If input is '1' then it adds $+\Delta$ step to the previous output. If input is '0' then one step 'Δ' is subtracted from the delayed signal. The LPF smoothens the staircase signal to reconstruct original message signal $x(t)$.

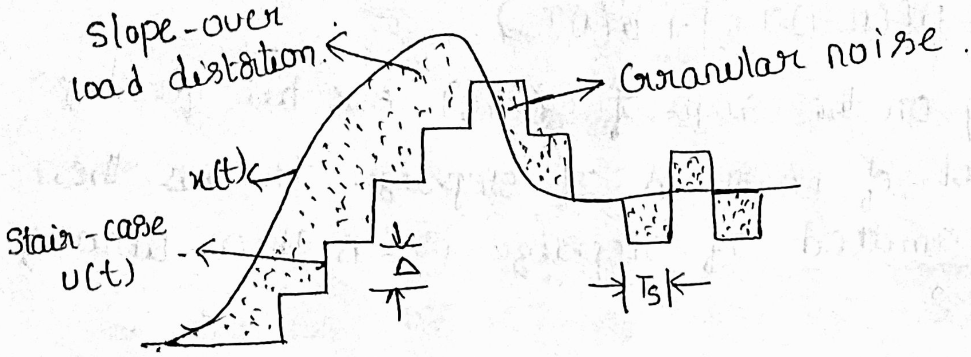
Advantages of DM :-

1. Since, DM transmits only one bit for one sample, the signaling rate and transmission channel bandwidth is quite small.
2. The Tx and Rx implementation for DM is very simple.

Drawbacks of DM :-

The DM has two major drawbacks as under.

1. Slope overload distortion
2. Granular or idle noise.



i) Slope overload distortion :-

The distortion arises because of large dynamic range of the input signal. From the figure, the rate of rise of input signal x(t) is so high that staircase signal cannot approximate it. Hence, there is a large error between the staircase approximated. and the original input signal, x(t). This error (or) noise is called as "slope-overload distortion".

To minimize this error, step size (Δ) must be increased when slope of signal x(t) is high. Since step size of DM remains fixed. its maximum or minimum slope occurs along straight lines. Therefore this modulator is known as " linear delta modulator.

ii) Granular (or) Idle noise :-

This noise occurs when step size is too large compared to small variations in the input signal. It means. for very small variations in the input signal, the staircase signal is changed by large amount (Δ). From the figure, when the input signal is almost flat, u(t) keeps on oscillating by ±Δ around the signal.

The error between input and approximated signal is called as granular noise. To overcome this problem, step size should be maintained low.

Bit Rate (sampling rate) of DM :-

Delta modulation bit rate (r) = No. of bits transmitted / sec.

= No. of samples / sec × No. of bits / sample

= $f_s \times 1$

= f_s.

The DM bit rate is $(1/N)$ times the bit rate of a PCM system, where 'N' is the number of bits per transmitted PCM codeword. From this, we can say that channel bandwidth required for DM is reduced to great extent when compared to that of a PCM system.

Example :- Given a sinewave of frequency f_m and amplitude A_m applied to a DM with step size 'Δ', show that slope overload distortion will occur if $A_m > \frac{\Delta}{2\pi f_m T_s}$, here T_s is sampling period.

Solution :- let us consider a sinewave is given as

$$x(t) = A_m \sin(2\pi f_m t).$$

The maximum slope of delta modulator may be given as.

$$\text{max. slope} = \frac{\text{Step size}}{\text{sampling period}} = \frac{\Delta}{T_s}.$$

we know that, slope over load distortion will take place if slope of $x(t)$ is greater than slope of DM.

$$\max\left|\frac{d}{dt}x(t)\right| > \frac{\Delta}{T_s}.$$

$$\max \left| \frac{d}{dt} A_m \sin(2\pi f_m t) \right| > \frac{\Delta}{T_S}.$$

$$\max \left| A_m \cos(2\pi f_m t)\, 2\pi f_m \right| > \frac{\Delta}{T_S}$$

$$A_m 2\pi f_m > \frac{\Delta}{T_S} \left[\because \max.\ \text{of}\ \cos\theta = 1\right]$$

$$\boxed{A_m > \frac{\Delta}{T_S(2\pi f_m)}}$$

Hence proved.

Evaluation of maximum output signal to noise ratio :-

we Know that the condition to avoid the slope over load distortion is expressed as.

$$A_m \leq \frac{\Delta}{2\pi f_m T_S}.$$

where A_m is peak amplitude of the sinusoidal signal

Δ is step size . f_m is maximum frequency of signal.

T_S is Sampling period.

$$\therefore A_m = \frac{\Delta}{2\pi f_m T_S}.$$

Therefore, the maximum value of the output signal power is expressed as

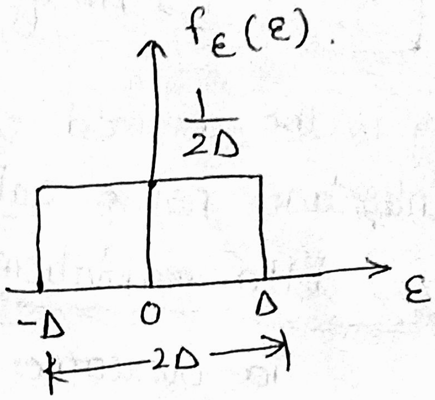
$$\text{signal power} = \frac{V_{rms}^2}{R} \left[\because R = 1\right]$$

$$V_{rms} = \frac{A_m}{\sqrt{2}}$$

$$P_{max} = \frac{A_m^2}{2}$$

$$\therefore \quad P_{max} = \frac{\Delta^2}{4\pi^2 f_m^2 T_s^2} \cdot \frac{1}{2} = \frac{\Delta^2}{8\pi^2 f_m^2 T_s^2}$$

Now, we require to obtain the expression for quantization noise power. The quantization error in DM lie in the interval $(-\Delta, \Delta)$. This error can be assumed to be uniformly distributed as shown in the figure.



$$f_\varepsilon(\varepsilon) = \begin{cases} 0 & t < -\Delta \\ \frac{1}{2\Delta} & -\Delta < t < \Delta \\ 0 & t > \Delta \end{cases}$$

$$E[\varepsilon^2] = \int_{-\infty}^{\infty} \varepsilon^2 f_\varepsilon(\varepsilon)\, d\varepsilon$$

$$= \int_{-\Delta}^{\Delta} \varepsilon^2 \cdot \frac{1}{2\Delta}\, d\varepsilon = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} \varepsilon^2\, d\varepsilon$$

$$= \frac{1}{2\Delta}\left[\frac{\varepsilon^3}{3}\right]_{-\Delta}^{\Delta} = \frac{1}{2\Delta}\left[\Delta^3 - (-\Delta^3)\right]$$

$$= \frac{1}{6\cancel{\Delta}} \times 2\cancel{\Delta}^3 = \frac{\Delta^2}{3}.$$

The DM signal is passed through a reconstruction LPF at output of a DM receiver. The bandwidth of LPF is ω such that $F_m \geq f_m$ and $F_m << f_s$ Normalized power at the filter output,

$$N = \frac{F_m}{f_s} \times \frac{\Delta^2}{3} = \frac{F_m T_s \Delta^2}{3} \quad \left(\because T_s = \frac{1}{f_s}\right)$$

Hence signal to noise ratio is given as,

$$\frac{S}{N} = \frac{\text{signal power (normalized)}}{\text{Noise power (normalized)}}$$

$$= \frac{\cancel{\Delta^2}}{8\pi^2 f_m^2 T_S^2} \times \frac{3}{\cancel{\Delta^2} F_M T_S}$$

$$\boxed{\frac{S}{N} = \frac{3}{8\pi^2 f_m^2 F_M T_S^3}}$$

This is the desired expression for the output signal to quantization noise ratio.

Adaptive delta modulation :-

To overcome the quantization errors due to slope overload and granular noise, the step size (Δ) is made adaptive to variations in input signal $x(t)$. If the i/p is varying slowly, the step size is reduced. This method is known as Adaptive delta modulation (ADM).

Transmitter part :-

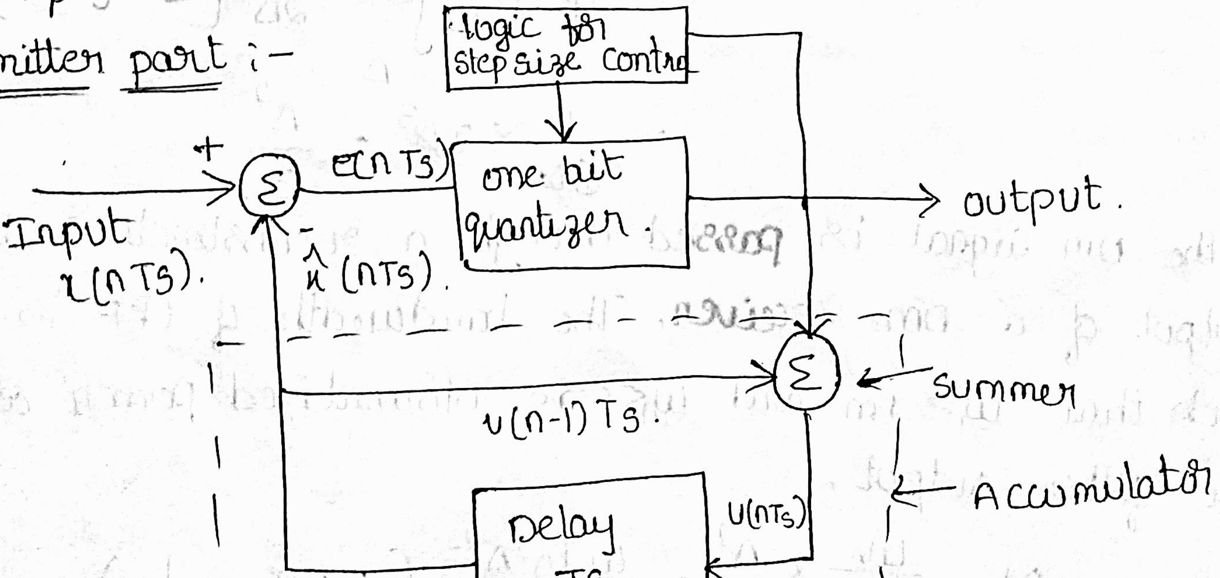


fig. Shows the transmitter part of ADM. The logic for step-size control is added to diagram.

The step size may increase or decrease according to specified rule depending on one bit quantizer output. If quantizer output is high, then step size may be doubled for next sample. If quantizer output is low, then step size may be reduced by one step.

Receiver part :-

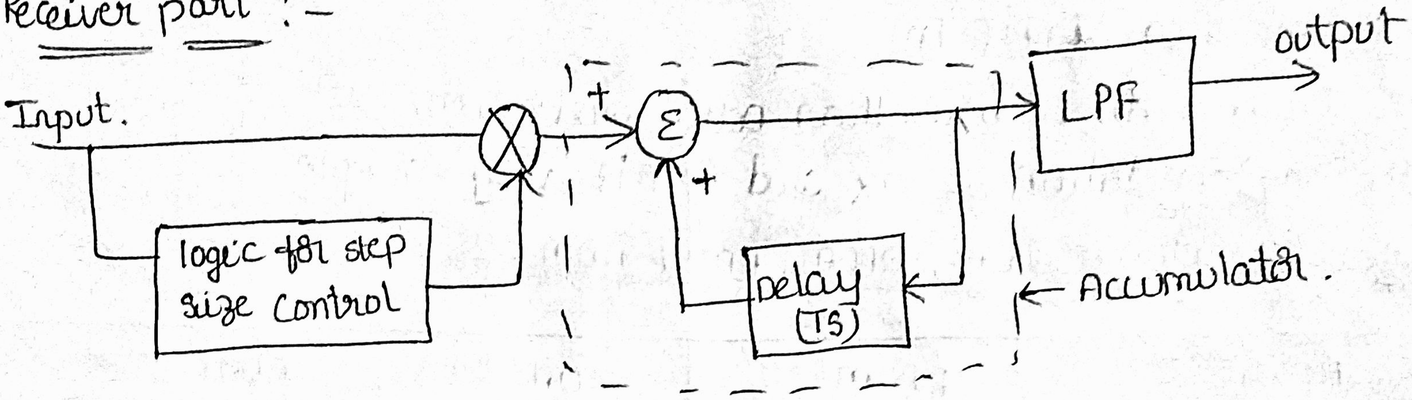
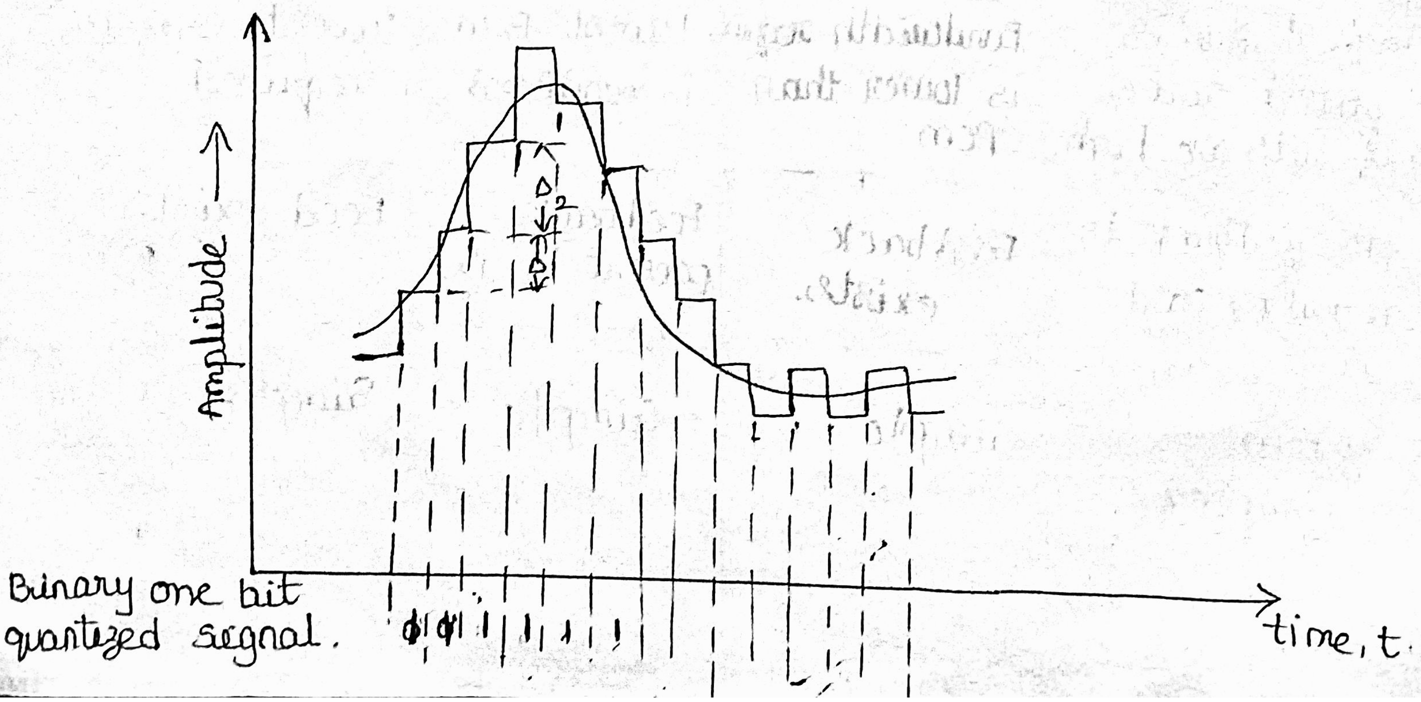


Fig. shows the receiver part of ADM. There are two portions. The first portion produces the step size from each incoming bit. Exactly, the same process is followed as that in transmitter. The previous input and present input decides stepsize. It is then applied to an accumulator which forms stair case waveform. Then LPF smoothens out the staircase waveform to reconstruct the original signal.



Advantages of ADM :-

1. The signal to noise ratio becomes better than ordinary DM because of reduction in slope overload distortion and idle noise.
2. Because of variable step size, dynamic range of ADM is wider than simple DM.
3. Bandwidth is better than Delta modulation.
4. Implementation of Tx and Rx is very simple

difference between PCM, DPCM, DM & ADM :-

PCM	DPCM.	DM	ADM.
1. It can use 4,8,16 bits / sample.	Bits can be more than one but less than PCM.	It uses only one bit per sample.	only one bit is used to encode one sample.
2. No. of levels depends on no. of bits level size is fixed.	Here Fixed no. of levels are used.	step size is fixed and cannot be varied.	According to signal variation, step size varies.
3. Error depends on no. of levels used.	slope over load distortion and quantization noise are present.	Slope over load & granular noise are present	Quantization noise is present but other errors are absent
4. highest B.W is required since no. of bits are high.	Bandwidth required is lower than PCM.	Lowest B.W is required.	lowest B.W is required.
5. No. feedback in Tx and Rx part.	Feedback exists.	Feedback present in Tx.	Feed exists.
6. system complex	Simple	Simple	Simple.

Digital Modulation Techniques

→ modulation is defined as the process by which some characteristic of a carrier is varied in accordance with a modulating signal.

→ In digital communications, the modulating signal consists of binary data. The data is used to modulate a carrier wave is usually sinusoidal with fixed frequency.

There are basically two types of transmission of digital signals.

Base band transmission :-

The digital data is transmitted over the channel directly. There is no carrier or any modulation. This is suitable for transmission over short distances.

Passband data transmission :-

The digital data modulates high frequency sinusoidal carrier. Hence it is also called digital c.w modulation. It is suitable for transmission over long distances.

Types of passband modulation :-

1. Amplitude shift keying :- Amplitude of the carrier is switched depending on the input digital signal, then it is called Amplitude shift keying.
2. phase shift keying :- phase of the carrier is switched depending on the input digital signal, then it is called phase shift keying.
3. Frequency shift keying :- Frequency of the carrier is switched depending on input digital signal, then it is called frequency shift keying.

→ The phase and frequency shift keying has constant amplitude. Because of constant amplitude of FSK & PSK the effect of non-linearities, noise interference is minimum on signal detection.

Types of Reception for Passband transmission :-

There are two types of methods for detection of passband signals.

Coherent (synchronous) detection :- In this method, the local carrier generated at the receiver is phase located with the carrier at the transmitter. Hence it is also called synchronous detection.

Non-coherent detection :- In this method, the receiver carrier need not be phase locked with transmitter carrier. Hence it is also called envelope detection. Non-coherent detection is simple but it has higher probability of error.

Requirements of Passband Transmission scheme :-

i) maximum data rate.
ii) minimum probability of symbol error.
iii) minimum transmitted power.
iv) maximum channel bandwidth.
v) maximum resistance to interfering signals.
vi) minimum circuit complexity.

Advantages of passband Transmission :-

1. long distance transmission.
2. Problems such as ISI and crosstalk are absent.
3. Passband transmission can take place over wireless channels also.
4. Large number of modulation techniques are available.

Drawbacks :-

1. It is not suitable for short distance communication.

Coherent binary amplitude shift keying (OR) ON-OFF keying :-

Amplitude shift keying (ASK) or ON-OFF keying (OOK) is the simplest digital modulation technique. In this method, there is only one unit energy carrier and it is switched on or off depending upon the input binary sequence.

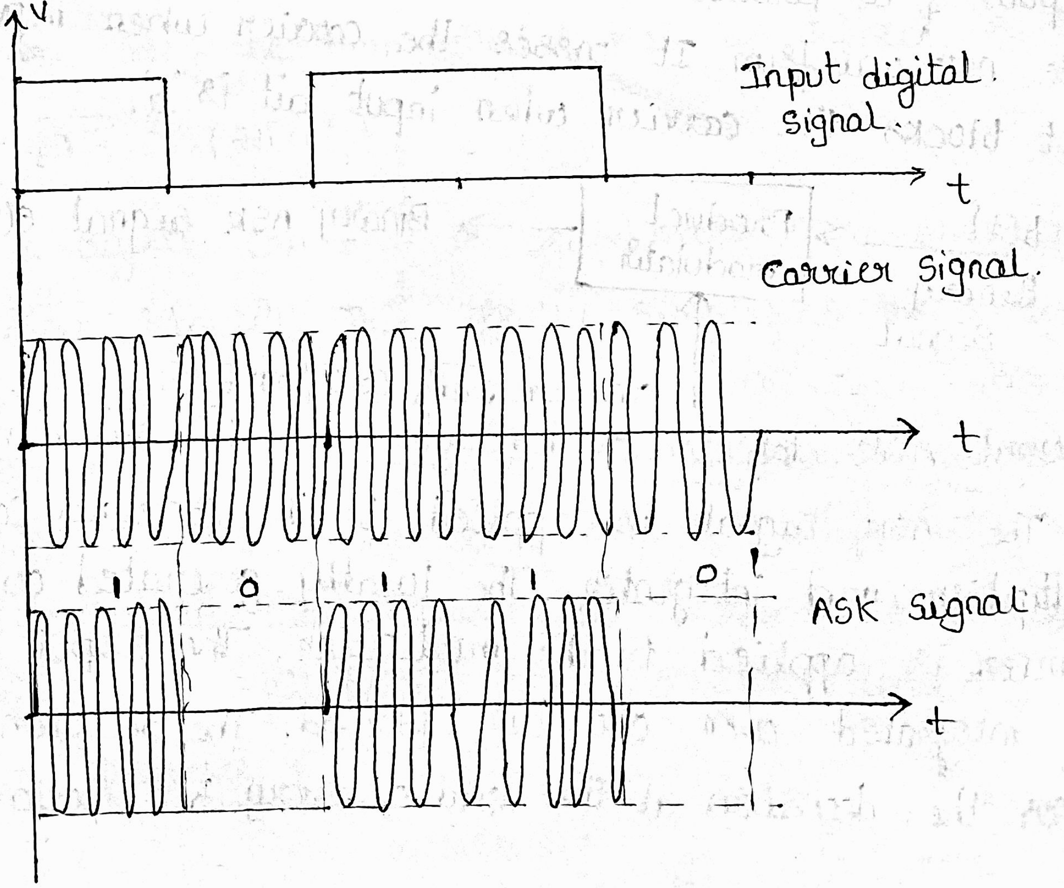
Expression and waveforms :-

The ASK waveform can be represented as.

$s(t) = \sqrt{2P_s}\,\cos(2\pi f_c t)$ (for transmit '1')

$s(t) = 0$ (for transmit '0').

To Transmit symbol '0', the signal $s(t) = 0$. no signal is transmitted. signal $s(t)$ contains some complete cycles of carrier frequency 'f_c'. Hence, the ASK wave form looks like an ON-OFF of the signal. Therefore, it is also known as the ON-OFF keying (OOK).

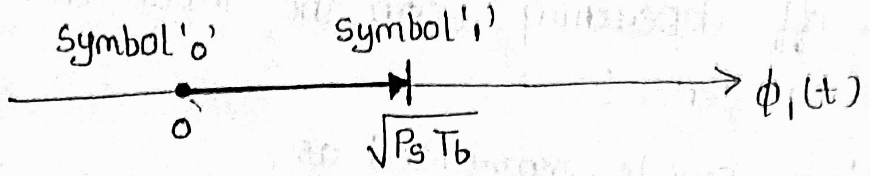


Signal space diagram of ASK :-

The ASK waveform of equation for symbol '1' can be represented as,

$$s(t) = \sqrt{P_s T_b} \cdot \sqrt{2/T_b} \cos(2\pi f_c t) = \sqrt{P_s T_b}\, \phi_1(t)$$

This means that there is only one carrier function $\phi_1(t)$. The signal space diagram will have two points on $\phi_1(t)$. One will be at zero and other will be at $\sqrt{P_s T_b}$.



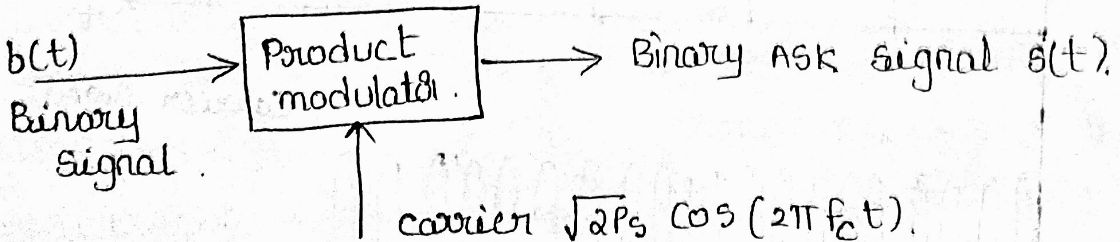
Signal space diagram of ASK.

Thus, the distance between the two signal points is.

$$d = \sqrt{P_s T_b} = \sqrt{E_b}.$$

Generation of ASK signal :-

ASK signal may be generated by simply applying the incoming binary data and the sinusoidal carrier to the two inputs of a product modulator. The resulting output will be the ASK waveform. It passes the carrier when input bit is '1'. It blocks the carrier when input bit is '0'.



Coherent ASK detector or demodulation of Binary ASK signal;-

The ASK signal is applied to the correlator consisting of multiplier and integrator. The locally generated coherent carrier is applied to the multiplier. The output of multiplier is integrated over one bit period. The decision device takes the decision at the end of every bit period.

It compares the output of integrator with the threshold. Decision is taken in favour of '1' when threshold is exceeded. Decision is taken as '0' if threshold is not exceeded.

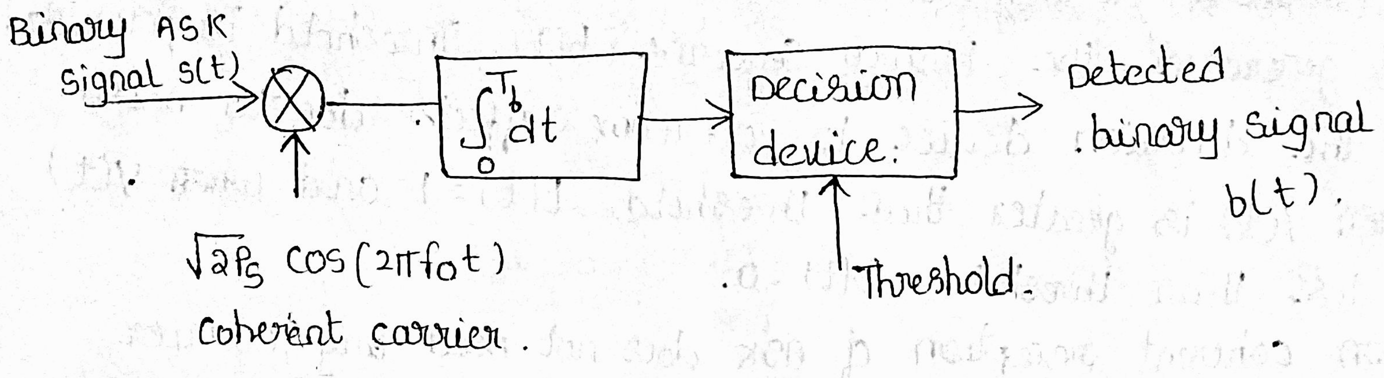


Fig. Block diagram of coherent ASK detector.

Non-coherent ASK detector :-

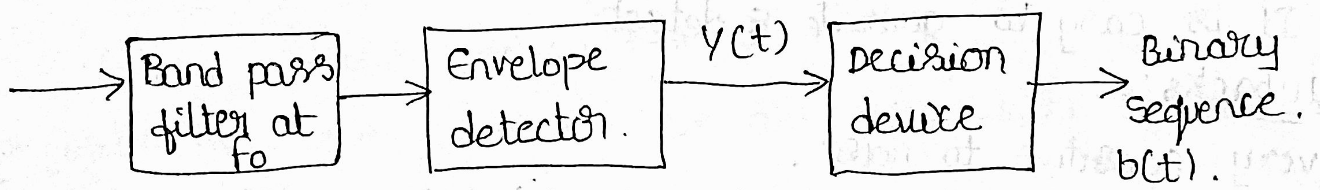


fig. Block diagram of non-coherent ASK detector.

In binary ASK case, the transmitted signal is defined as.

$$s(t) = \sqrt{2P_s}\cos(2\pi f_c t).$$

Binary ASK signal can also be demodulated non-coherently using envelope detector. This greatly simplifies the design consideration required in synchronous detection. Non-coherent detection schemes do not require a phase-coherent local oscillator. This method involves some form of rectification and low pass filtering at the receiver. The received ASK signal is given to band pass filter. This band pass filter passes only carrier frequency, f

→ The envelope detector generates high output voltage when carrier is present. when carrier is absent, there is only noise at the input of envelope detector. The decision device is basically a regenerator.

→ It generates the binary sequence b(t). Threshold is provided to the decision device to overcome effects due to noise when y(t) is greater than threshold b(t) = 1 and when y(t) is less than threshold. b(t) = 0.

→ Non-coherent reception of ASK does not need any carrier synchronization.

Features:-

1. BASK is simple.
2. It is easy to generate & detect.

Drawbacks:-

1. very sensitive to noise.
2. It finds limited application in data transmission.
3. It is used at very low bit rates, upto 100 bits per sec.

Binary Phase Shift Keying :- (BPSK)

Binary phase shift keying is used for high bit rates. In BPSK phase of the sinusoidal carrier changed according to the data bit to be transmitted.

Expression for BPSK :-

In a binary phase shift keying (BPSK), the binary symbols '1' and '0' modulate the phase of the carrier the carrier is

$$s(t) = A\cos(2\pi f_c t).$$

A is represents peak value of sinusoidal carrier. For the standard 1Ω load resistor.

The power dissipated will be,

$$P = \frac{1}{2} A^2$$

$$A = \sqrt{2P}.$$

When the symbol is changed, then the phase of the carrier is changed by 180 degrees. for example.

for symbol '1'

$$S_1(t) = \sqrt{2P} \cos(2\pi f_c t).$$

for symbol '0'

$$S_2(t) = \sqrt{2P} \cos(2\pi f_c t + \pi).$$

$$\therefore \cos(\theta + \pi) = -\cos\theta.$$

$$S_2(t) = -\sqrt{2P} \cos(2\pi f_c t).$$

with the above equations, we can define BPSK signal combinely as,

$$S(t) = b(t) \sqrt{2P} \cos(2\pi f_c t)$$

where $b(t) = +1$ when binary '1' is to be transmitted.

$= -1$ when binary '0' is to be transmitted.

Generator of BPSK signal :-

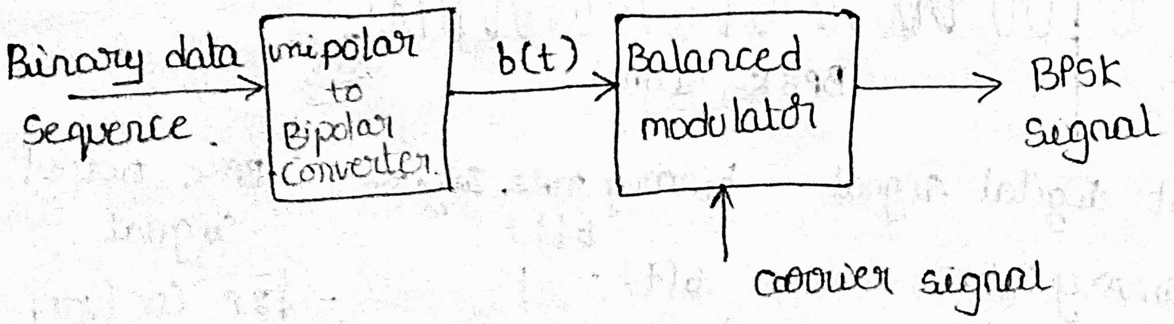
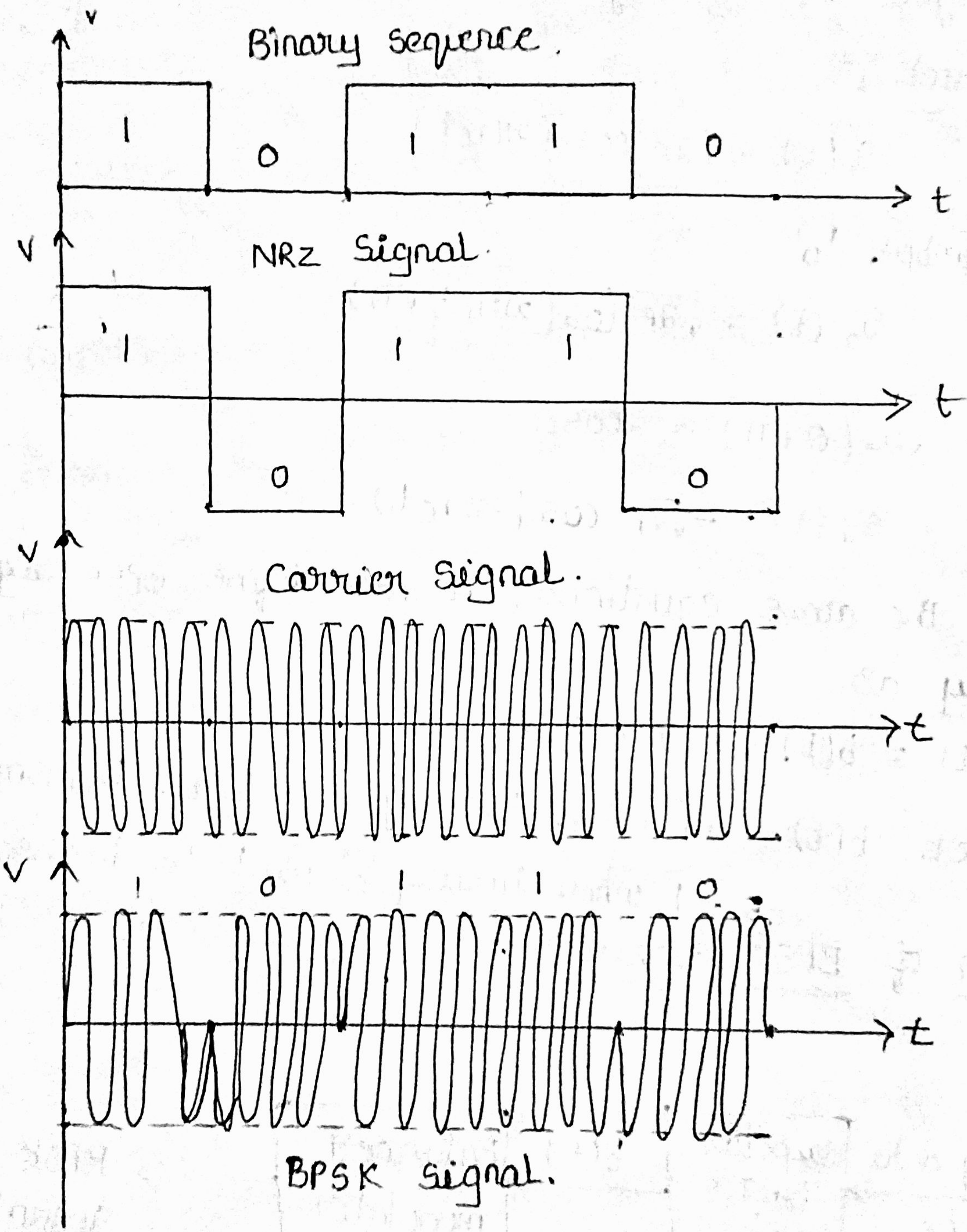


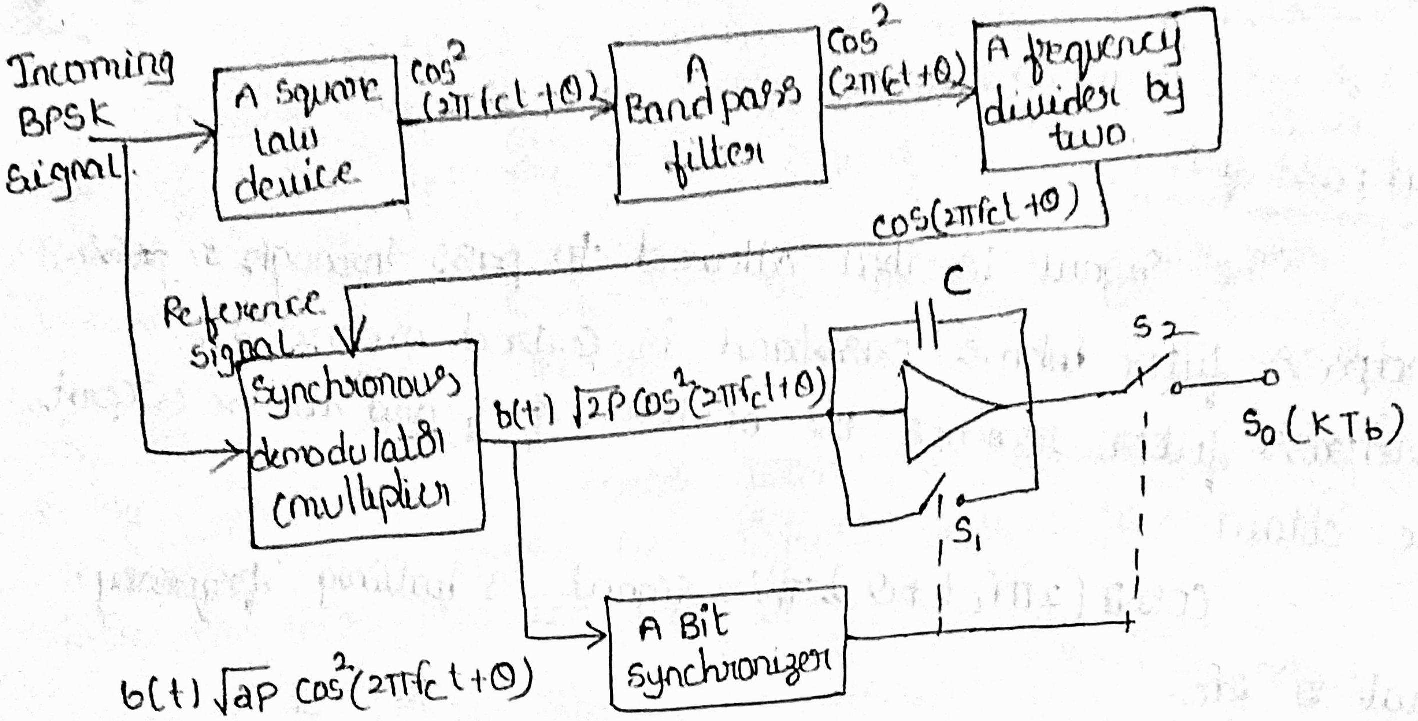
fig - Block diagram of Generation of BPSK signal

BPSK Signal is generated by applying carrier signal to a balanced modulator. The binary data signal (0's and 1's) is converted into a NRZ bipolar signal by an NRZ encoder. Here the bipolar signal b(t) is applied as a modulating signal to the balanced modulator. A NRZ level encoder converts the binary data sequence into bipolar NRZ signal.



input digital signal	binary NRZ signal b(t)	BPSK output signal
Binary '0'	b(t) = -1	$-\sqrt{2P}\cos(2\pi f_c t)$
Binary '1'	b(t) = 1	$+\sqrt{2P}\cos(2\pi f_c t)$.

Coherent PSK Signal :-



Coherent detection of PSK signal

The input signal undergoes the phase change depending upon the time delay from transmitter to receiver. This phase change is normally fixed phase shift in the transmitted signal let the phase shift is θ. the signal at the input of the receiver is

$$s(t) = b(t)\sqrt{2P}\cos(2\pi f_c t + \theta).$$

Square law device :-

The received signal is allowed to pass through a square law device. The output of the square law device the signal will be $\cos^2(2\pi f_c t + \theta)$.

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Therefore, we have.

$$\cos^2(2\pi f_c t + \theta) = \frac{1+\cos(2\pi f_c t + \theta)}{2}$$

$= \frac{1}{2} + \frac{1}{2} \cos 2(2\pi f_c t + \theta)$

$\uparrow$
DC level.

Band pass filter :-

The signal is then allowed to pass through a pass bandpass filter whose passband is centred around 2fc. Bandpass filter removes the DC level of $\frac{1}{2}$ and at the output, we obtain

$\cos 2(2\pi f_c t + \theta)$. This signal is having frequency equal to 2fc.

Frequency divider :- The signal is passed through a frequency divider by two. Thus at the output of frequency divider, we get a carrier signal whose frequency is fc that is $\cos(2\pi f_c t + \theta)$.

Synchronous demodulator :-

The synchronous demodulator multiplies the input signal and the recovered carrier. Therefore at the output of multiplier we get,

$= b(t)\sqrt{2P} \cos(2\pi f_c t + \theta) \times \cos(2\pi f_c t + \theta)$

$= b(t)\sqrt{2P} \cos^2(2\pi f_c t + \theta)$

$= b(t)\sqrt{2P} \times \frac{1}{2}[1 + \cos 2(2\pi f_c t + \theta)]$

$= b(t)\sqrt{\frac{P}{2}}[1 + \cos 2(2\pi f_c t + \theta)]$.

Bit synchronizer and integrator :-

The signal is then applied to the bit synchronizer and integrator. The integrator integrates the signal over one bit period The bit synchronizer takes care of starting and ending times of a bit.

1. At the end of bit duration T_b, the bit synchronizer closes switch S_2 temperorily. This connects the output of an integrator to the decision device.
2. The synchronizer then opens switch S_2 and switch S_1 is closed temperorily. This resets the integrator voltage to zero. The integrator then integrates next bit.

Also, in the K^{th} bit interval, we can write output signal.

$$S_0(KT_b) = b(KT_b)\sqrt{\frac{P}{2}}\left[\int_{(K-1)T_b}^{KT_b} 1\,dt + \int_{(K-1)T_b}^{KT_b} \cos 2(2\pi f_c t + \theta)\,dt\right]$$

Here $\int_{(K-1)T_b}^{KT_b} \cos 2(2\pi f_c t + \theta)\,dt = 0$, since average value of sinusoidal waveform is zero. If integration is done over full cycles.

$$S_0(KT_b) = b(KT_b)\sqrt{\frac{P}{2}}\int_{(K-1)T_b}^{KT_b} 1\,dt.$$

$$= b(KT_b)\sqrt{\frac{P}{2}}\,[t]_{(K-1)T_b}^{KT_b}$$

$$= b(KT_b)\sqrt{\frac{P}{2}}\,[KT_b - (K-1)T_b]$$

$$= b(KT_b)\sqrt{\frac{P}{2}}\,[KT_b - KT_b + T_b$$

$$\boxed{S_0(KT_b) = b(KT_b)\sqrt{\frac{P}{2}}\,T_b}$$

This equation shows that the output of the receiver depends on input. This signal is then given to a decision device which decides whether transmitted symbol was zero or one.

A Geometrical Representation for BPSK Signals :-

The BPSK signal carries the information about two symbols. These symbols are symbol '1' and symbol '0'.

$$s(t) = \sqrt{2P} \cos(2\pi f_c t) . b(t).$$

Rearrange the last equation as

$$s(t) = b(t) . \sqrt{PT_b} . \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

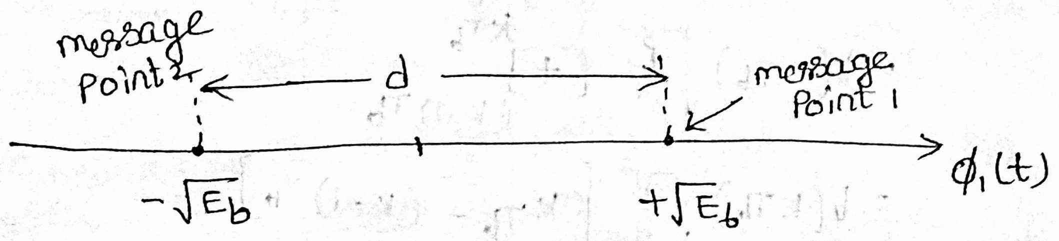
$$s(t) = b(t) \sqrt{PT_b} . \phi_1(t) \quad \therefore \phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

The bit Energy E_b is defined in terms of Power 'P' and one bit duration T_b as,

$$E_b = PT_b \quad , \quad b(t) = \pm 1$$

$$s(t) = \pm \sqrt{E_b} \, \phi_1(t).$$

Thus, on the singe axis of $\phi_1(t)$, there will be two points. one point will be located at $+\sqrt{P.T_b}$ or $\sqrt{E_b}$ and other will be located at $-\sqrt{PT_b}$ or $-\sqrt{E_b}$.



Representation of BPSK Signal.

At the receiver end, the point at $+\sqrt{E_b}$ on $\phi_1(t)$ represents symbol '1' and point at $-\sqrt{E_b}$ represents symbol '0'. This separation is generally called distance 'd'.

$$d = \sqrt{E_b} - (-\sqrt{E_b}) = 2\sqrt{E_b}.$$

distance 'd' increases, the isolation between the symbols in BPSK signal is more, Thus, probability of error reduces.

Bandwidth for BPSK signal :-

The spectrum of the BPSK signal is centred around the carrier frequency fc.

If $T_b = \frac{1}{f_b}$, then $f_b = \frac{1}{T_b}$, for BPSK, the maximum frequency in the base band signal will be f_b.

$$B.W = \text{highest frequency} - \text{lowest frequency.}$$

$$= f_c + f_b - (f_c - f_b)$$

$$= \cancel{f_c} + f_b - \cancel{f_c} + f_b$$

$$\boxed{B.W = 2f_b}$$

Drawbacks of BPSK :- In BPSK receiver to generate the carrier in the receiver, squaring $b(t)\sqrt{2P}\cos(2\pi f_c t + \theta)$. If the received signal is $-b(t)\sqrt{2P}\cos(2\pi f_c t + \theta)$ then the squared signal remains same as before.

Therefore, it is not possible to determine whether the received signal is equal to $b(t)$ or $-b(t)$. This results in Ambiguity in the output signal.

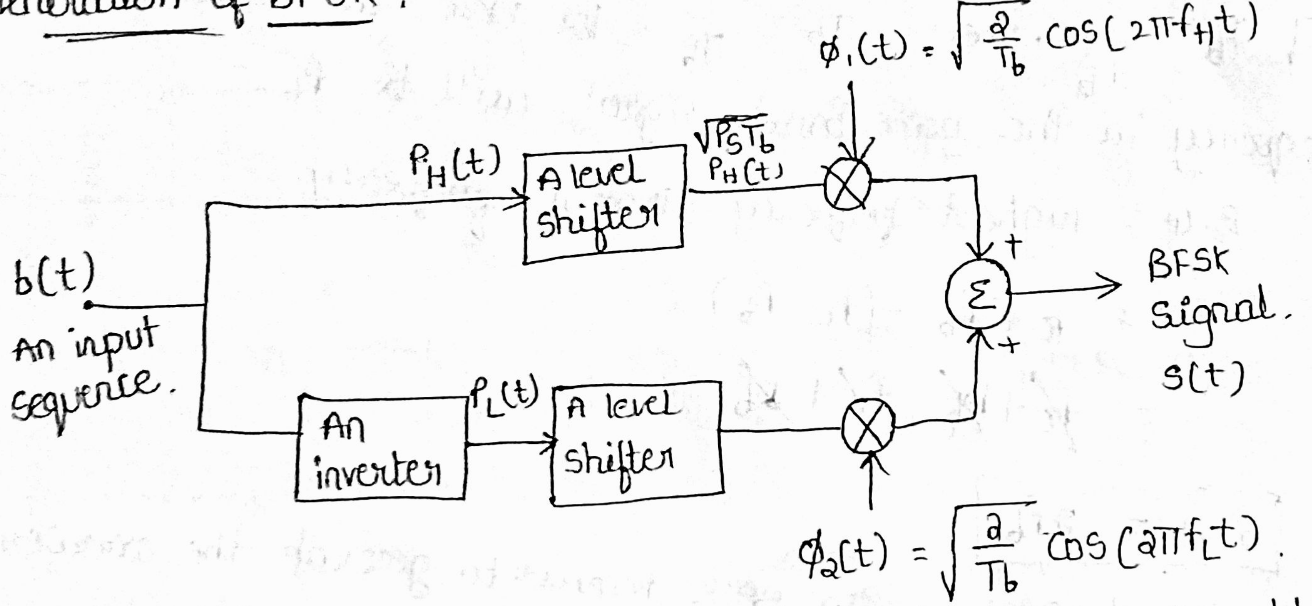
Binary Frequency shift keying (BFSK) :-

In binary frequency shift keying (BFSK), the frequency of a sinusoidal carrier is shifted according to the binary symbol. The frequency of a sinusoidal carrier is shifted between two discrete values. The phase of the carrier is unaffected. This means that we have two different frequency signals according to binary symbols.

If b(t) = 1, then $S_H(t) = \sqrt{2P_S}\cos(2\pi f_H t)$

If b(t) = 0, then $S_L(t) = \sqrt{2P_S}\cos(2\pi f_L t)$

Generation of BFSK :-



The input Sequence b(t) is same as $P_H(t)$. An inverter is added after b(t) to get $P_L(t)$. The level shifter $P_H(t)$ and $P_L(t)$ are unipolar signals. The level shifter converts the +1 level to $\sqrt{P_S T_b}$ zero level is unaffected.

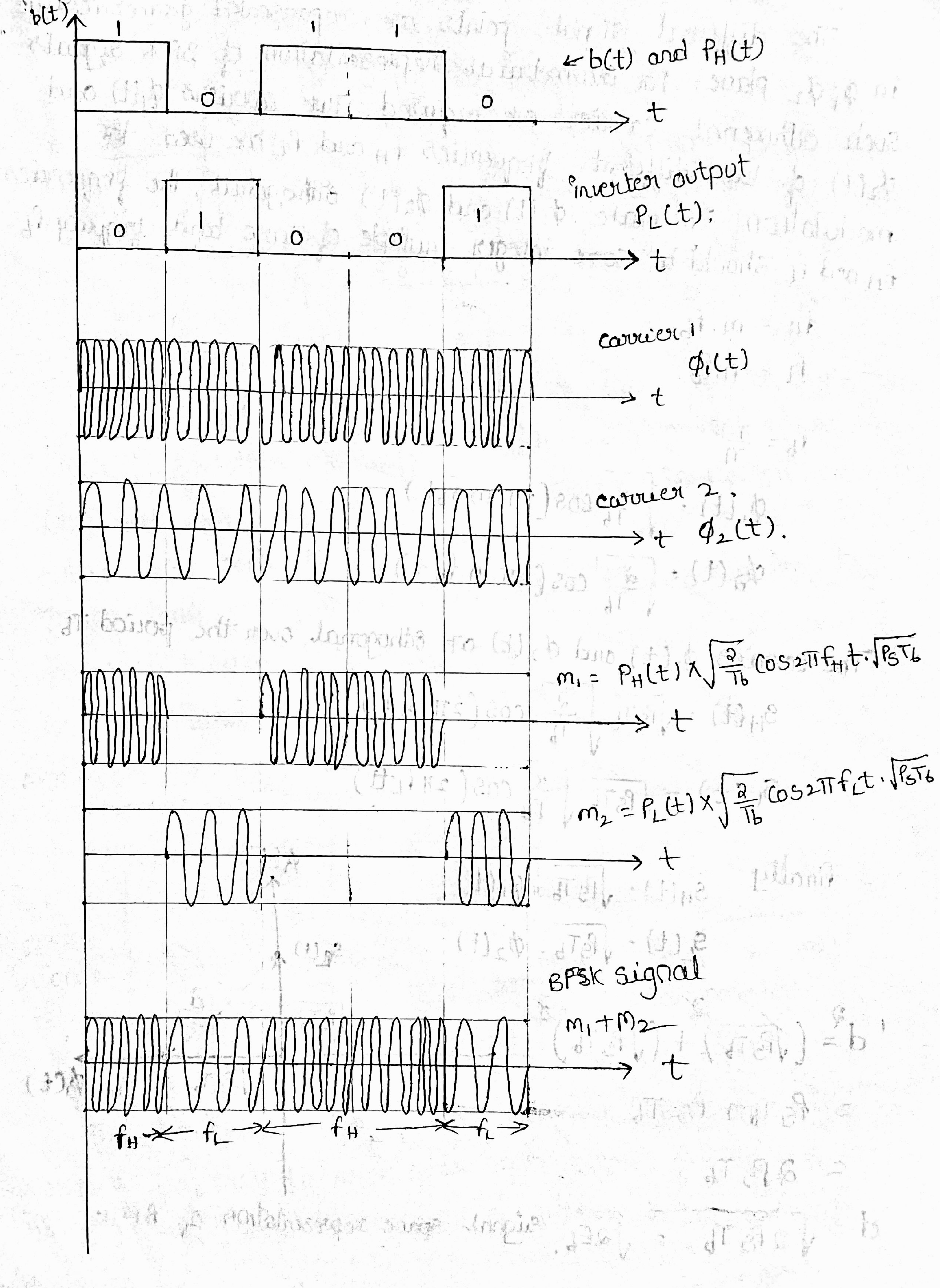
In other words, when a binary '0' is to be transmitted $P_L(t) = 1$ and $P_H(t) = 0$, and for a binary '1' is to be transmitted $P_H(t) = 1$ and $P_L(t) = 0$. Hence, the transmitted signal will have a frequency either f_H & f_L. The two carrier signals $\phi_1(t)$ and $\phi_2(t)$ are used. $\phi_1(t)$ and $\phi_2(t)$ are orthogonal to each other. In one bit period of input signal $\phi_1(t)$ or $\phi_2(t)$ have integral number of cycles.

$$s(t) = \sqrt{2P_S}\cos(2\pi f_H t)\,P_H(t) + \sqrt{2P_S}\cos(2\pi f_L t)\,P_L(t)$$

$$= \sqrt{P_S T_b}\,P_H(t)\sqrt{\frac{2}{T_b}}\cos(2\pi f_H t) + \sqrt{P_S T_b}\,P_L(t)\sqrt{\frac{2}{T_b}}\cos(2\pi f_L t)$$

$$= \sqrt{E_b}\sqrt{\frac{2}{T_b}}\,P_H(t)\cos(2\pi f_H t) + \sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_L t)$$

wave forms :-


b(t)
1
0
1
1
0
← b(t) and $P_H(t)$
t
0
1
0
0
1
inverter output
$P_L(t)$;
t
carrier 1
$\phi_1(t)$
t
carrier 2.
t
$\phi_2(t)$.
$m_1 = P_H(t) \times \sqrt{\frac{2}{T_b}} \cos 2\pi f_H t \cdot \sqrt{P_s T_b}$
t
$m_2 = P_L(t) \times \sqrt{\frac{2}{T_b}} \cos 2\pi f_L t \cdot \sqrt{P_s T_b}$
t
BPSK signal
$m_1 + m_2$
t
f_H
f_L
f_H
f_L

Geometrical Representation of Orthogonal BFSK :-

The different signal points are represented geometrically in $\phi_1 \phi_2$ plane. For Geometrical representation of BFSK signals such orthogonal carriers are required. Two carriers $\phi_1(t)$ and $\phi_2(t)$ of two different frequencies f_H and f_L are used for modulation. To make $\phi_1(t)$ and $\phi_2(t)$ orthogonal, the frequencies f_H and f_L should be some integer multiple of base band frequency f_b.

$$f_H = m.f_b$$
$$f_L = n.f_b.$$

$$f_b = \frac{1}{T_b}$$

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi\, m.f_b t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi. n. f_b\, t)$$

The carriers $\phi_1(t)$ and $\phi_2(t)$ are orthogonal over the period T_b.

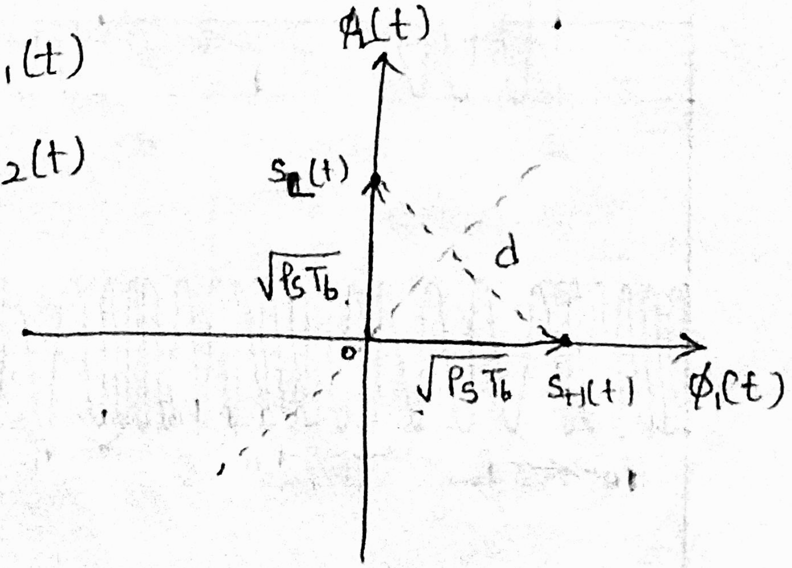
$$S_H(t) = \sqrt{P_S T_b} \sqrt{\frac{2}{T_b}} \cos(2\pi f_H t)$$

$$S_L(t) = \sqrt{P_S T_b} \sqrt{\frac{2}{T_b}} \cos(2\pi f_L t)$$

finally

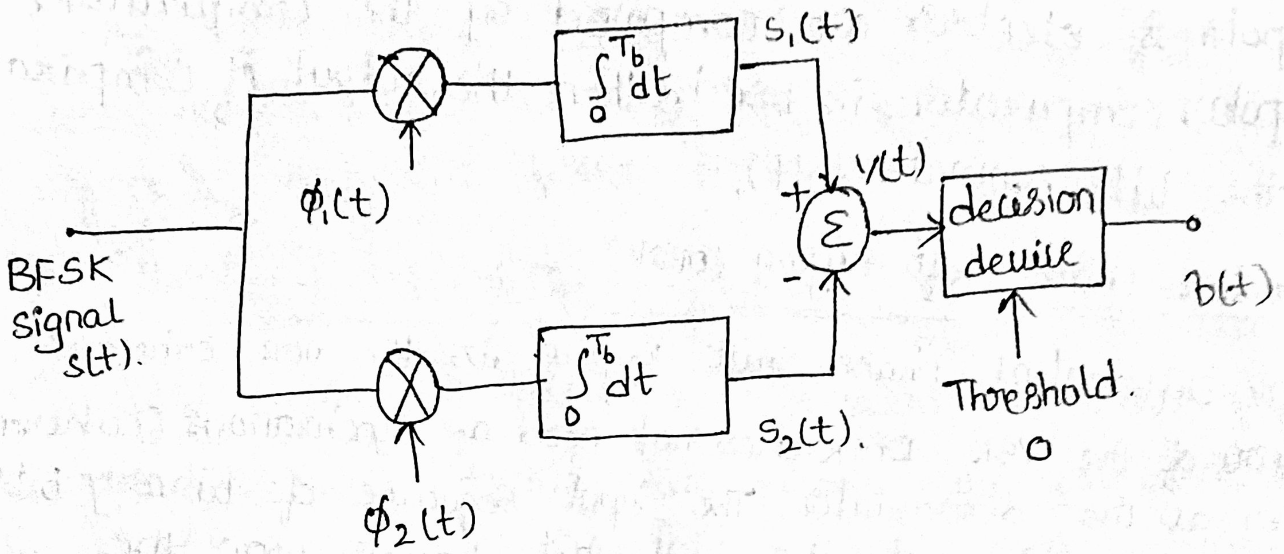
$$S_H(t) = \sqrt{P_S T_b} \cdot \phi_1(t)$$
$$S_L(t) = \sqrt{P_S T_b} \cdot \phi_2(t)$$

$$d^2 = \left(\sqrt{P_S T_b}\right)^2 + \left(\sqrt{P_S T_b}\right)^2$$
$$= P_S T_b + P_S T_b$$
$$= 2 P_S T_b$$



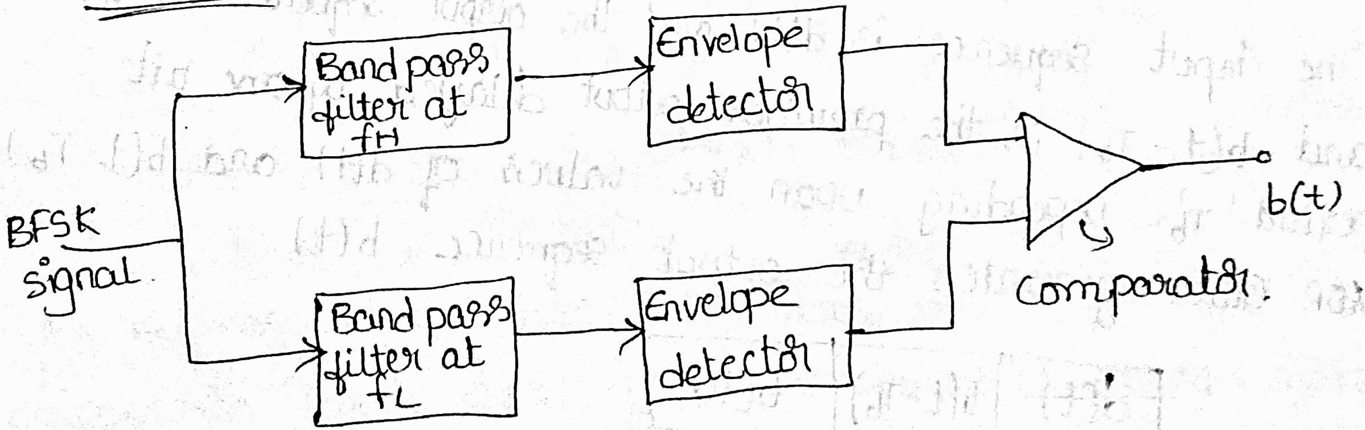
$$d = \sqrt{2 P_S T_b} = \sqrt{2E_b}$$ signal space representation of BFSK.

Coherent BFSK Receiver :-



There are two corelators for two frequencies of FSK signal. These corelators are supplied with locally generated carriers $\phi_1(t)$ and $\phi_2(t)$. If the transmitted frequency is f_H, then output $s_1(t)$ will be higher than $s_2(t)$. Hence $y(t)$ will be greater than zero. The decision device then decides in favour of binary '1'. If $s_2(t) > s_1(t)$ then $y(t) < 0$ and decision device decides in favour of 0.

Non-coherent BFSK Receiver :-



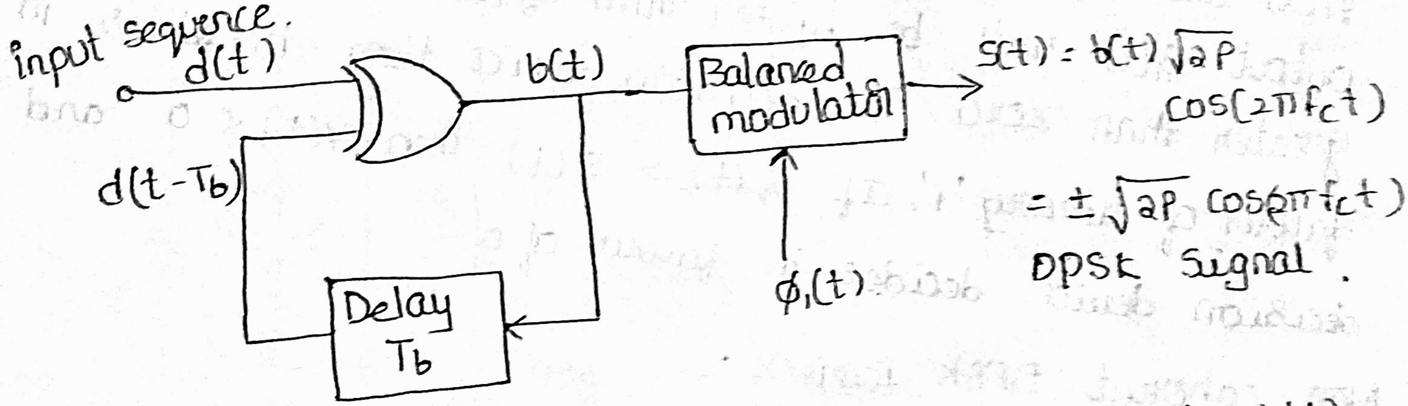
The receiver consists of two bandpass filters, one with carrier frequency f_H and other with centre frequency f_L. Since $f_H - f_L = 2f_b$. The outputs of filters do not overlap.

The outputs of filters are applied to envelop detectors. The outputs of detectors are compared by the comparator. If unipolar comparator is used, then the output of comparator is the bit sequence b(t).

Differential Phase Shift Keying (DPSK) :-

Differential phase shift keying is the non-coherent version of the PSK. DPSK does not need a synchronous (coherent) carrier at the demodulator. The input sequence of binary bits is modified such that the next bit depends upon the previous bit. Therefore, in the receiver, the previous received bits are used to detect the present bit.

Generation of DPSK :-



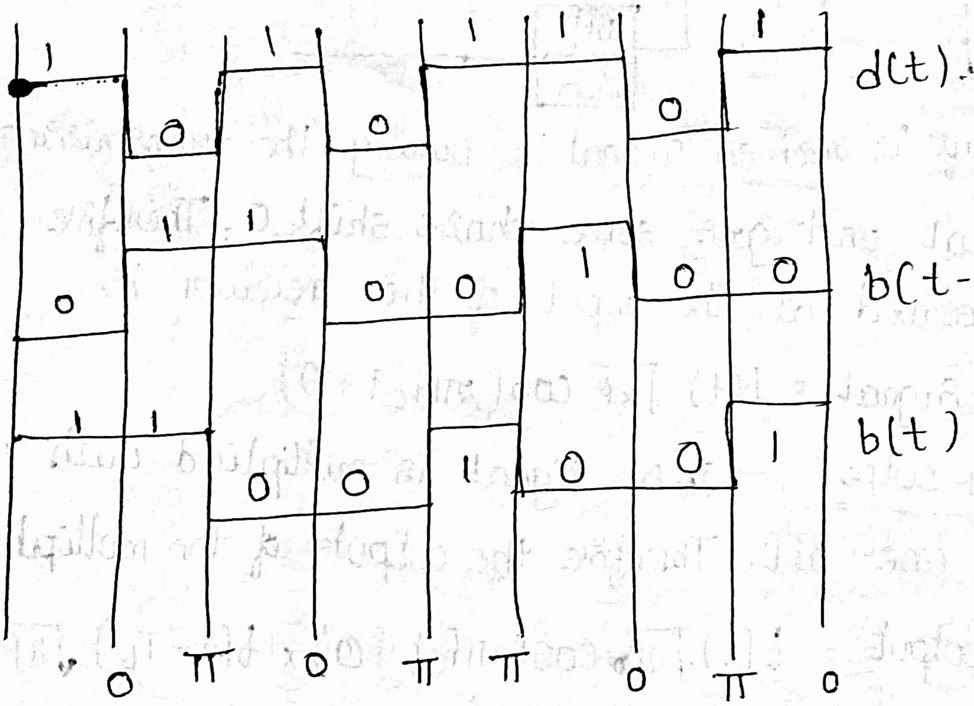
The input sequence is d(t) and the output sequence is b(t) and $b(t-T_b)$ is the previous output delayed by one bit period 'T_b'. Depending upon the values of d(t) and $b(t-T_b)$, XOR Gate generates the output sequence b(t).

d(t)	$b(t-T_b)$	b(t)
0 (-1V)	0 (-1V)	0 (-1V)
0 (-1V)	1 (1V)	1 (1V)
1 (1V)	0 (-1V)	1 (1V)
1 (1V)	1 (1V)	0 (-1V)

The data stream b(t) is applied to the input of the encoder. The output of the encoder is applied to one input of the product modulator. To the other input of this product modulator, a sinusoidal carrier of fixed amplitude and frequency is applied.

$$\boxed{b(t) = d(t) \oplus b(t-T_b)}$$

An aribitrary waveform d(t) is taken depending on this sequence, b(t) and b(t-Tb) are found. while drawing the waveforms the value of b(t-Tb) is not known initially in first interval. Therefore it is assumed to be zero and then waveforms are drawn.



when $d(t) = 0,\ b(t) = b(t-T_b)$

$d(t) = 1,\ b(t) = \overline{b(t-T_b)}$

→ The sequence b(t) modulates the sinusoidal carrier. when b(t) changes the level, phase of the carrier is changed since b(t) changes its level only when d(t) = 1. It shows that phase of the carrier is changed only if d(t) = 1

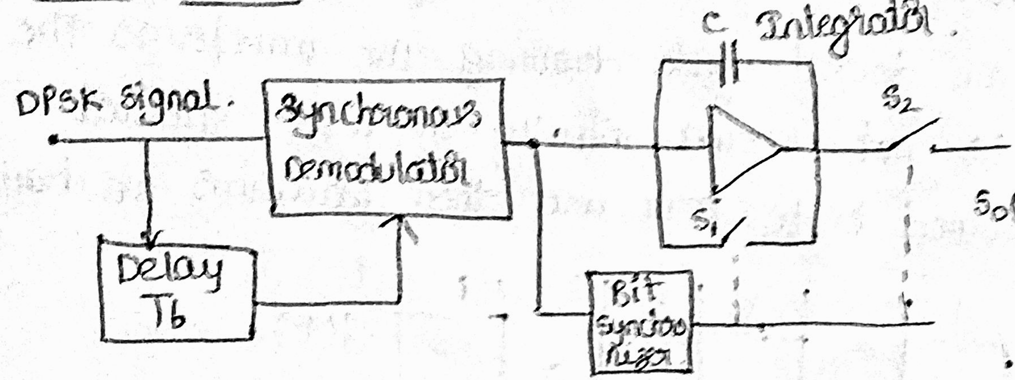
Always two successive bits are checked for any change of level. Hence onesymbol has two bits.

Symbol duration $(T) = 2T_b$

The modulator output is $s(t) = b(t)\sqrt{2P}\cos 2\pi f_c t$

$$s(t) = \pm\sqrt{2P}\cos 2\pi f_c t.$$

DPSK Receiver :-



1. phase shift in received signal :- During the transmission, the DPSK signal undergoes some phase shift θ. Therefore the signal received at the input of the receiver is.

Received signal $= b(t)\sqrt{2P}\cos(2\pi f_c t + \theta).$

2. multiplier output :- This signal is multiplied with its delayed version by one bit. Therefore the output of the multiplier is

multiplier output $= b(t)\sqrt{2P}\cos(2\pi f_c t + \theta) \times b(t - T_b)\sqrt{2P}$ $\cos[2\pi f_c(t - T_b) + \theta]$

$$= b(t)\, b(t - T_b)\,(2P)\cos(\underbrace{2\pi f_c t + \theta}_{A})\cos\underbrace{[2\pi f_c(t - T_b) + \theta]}_{B}$$

$$\therefore \cos(A).\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$= b(t)\, b(t - T_b)\,(2P)\frac{1}{2}\cos(2\pi f_c t + \theta)\cos[2\pi f_c t - 2\pi f_c T_b + \theta]$$

$$= b(t)\; b(t-T_b)\;(2P)\;\cos(2\pi f_c t+\theta)\;\cos(2\pi f_c t-2\pi f_c T_b+\theta]$$

$$= b(t)\; b(t-T_b)(\cancel{2}P)\;\frac{1}{\cancel{2}}\Big[\cos(2\pi \cancel{f_c t}+\cancel{\phi}-2\pi \cancel{f_c t}+2\pi f_c T_b-\cancel{\phi}]+\cos\big(2\pi f_c t+\theta+2\pi f_c t - 2\pi f_c T_b+\theta)\Big]$$

$$= b(t)\; b(t-T_b)\;P.\,[\cos(2\pi f_c T_b)+\cos(4\pi f_c t-2\pi f_c T_b+2\theta)]]$$

$$= b(t)\; b(t-T_b)\;P\Big\{\cos(2\pi f_c T_b)+\cos\Big[4\pi f_c\Big[t-\frac{T_b}{2}\Big]+2\theta\Big]\Big\}$$

f_c is the carrier frequency and T_b is one bit period. T_b contains integral number of cycles of f_c.

$$f_b=\frac{1}{T_b}. \qquad f_c=n f_b \Rightarrow f_o=\frac{n}{T_b}$$

multiplier output $= b(t)\; b(t-T_b)\;P\Big\{\cos 2\pi n+\cos\Big[4\pi f_c\Big[t-\frac{T_b}{2}\Big)+2\theta\Big]\Big\}$ $\therefore \cos 2\pi n = 1$

$$= b(t)\;b(t-T_b)\;P + b(t)\;b(t-T_b)\;P\cos\Big(4\pi f_c\Big(t-\frac{T_b}{2}\Big)+2\theta\Big]$$

Integrator :- The above Signal is given to the integrator. In the kth bit interval, the integrator output can be written as

$$s_o(KT_b) = b(KT_b)\; b[(K-1)T_b]\,P\int_{(K-1)T_b}^{KT_b} 1\, dt + b(KT_b)\; b[(K-1)T_b]\,P\int_{(K-1)T_b}^{KT_b}\Big[\cos\Big(4\pi f_c\Big(t-\frac{T_b}{2}\Big)+2\theta\Big]dt$$

The integration of the second term will be zero since it is integration of carrier over one bit duration. The carrier has integral number of cycles over one bit period hence integration is zero.

$$s_0(KT_b) = b(KT_b)\, b[(K-1)T_b]\, P\,[KT_b - (K-1)T_b]$$

$$= b(KT_b)\, b[(K-1)T_b]\, P\,[\cancel{K}T_b - \cancel{K}T_b + T_b].$$

$$= b(KT_b)\, b[(K-1)T_b]\, PT_b.$$

The product of $b(KT_b)$ and $b(K-1)T_b$ decides the sign of PT_b.

If $b(t)\, b(t-T_b) = 1V$ then $d(t) = 0$.

$\therefore$ $b(t)$ and $b(t-T_b)$ both are $+1V$ or $-1V$

If $b(t)\, b(t-T_b) = -1V$ then $d(t) = 1$

That is $b(t) = -1V$, $b(t-T_b) = +1V$ and vice versa.

Therefore $b(t)\, b(t-T_b) = -1$.

Decision device :-

$$s_0(KT_b) = b(KT_b)[b(K-1)T_b]\, PT_b.$$

$$\text{If } s_0(KT_b) = \begin{cases} -PT_b, \text{ then } d(t) = 1 \text{ and} \\ +PT_b, \text{ then } d(t) = 0. \end{cases}$$

Bandwidth of DPSK :-

One previous bit is always used to define the phase shift in next bit, the symbol can be said to have two bits. Therefore one symbol duration (T) is equivalent to two bits duration $(2T_b)$.

∴ Symbol duration $T = 2T_b$

Bandwidth $B.W = \frac{2}{T}$

$= \frac{2}{2T_b}$

$= \frac{1}{T_b}$

$$\boxed{B.W = f_b}$$

Thus the minimum bandwidth in DPSK is equal to f_b, that is maximum baseband signal frequency.

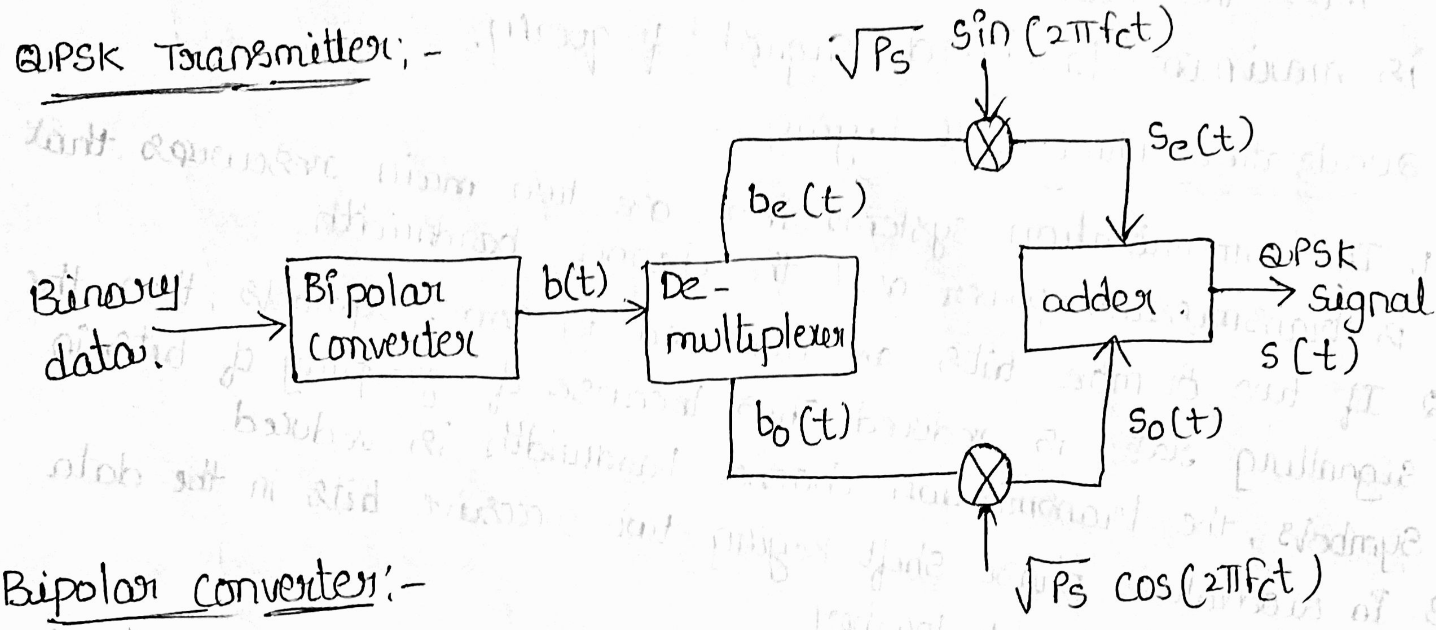
Quadrature Phase Shift Keying :-

1. In communication systems there are two main resources that is transmission power and the channel bandwidth.
2. If two or more bits are combined in some symbols, then the signalling rate is reduced. Thus because of grouping of bits in symbols, the transmission channel bandwidth is reduced.
3. In quadrature phase shift keying. two successive bits in the data sequence are grouped together.
4. In BPSK we known that when symbol changes the level, the phase of the carrier is changed by 180°. since there were only two symbols in BPSK, the phase shift occurs in two levels only.
5. In QPSK two successive bits are combined. This combination of two bits forms four distinct symbols. when the symbol is changed to next symbol the phase of the carrier is changed.

Symbol and corresponding phase shifts in QPSK :-

Input successive bits		Symbol	Phase shift in carrier	
0 (-1)	0 (-1V)	S_1	$3\pi/4$	135°
0 (-1V)	1 (1V)	S_2	$5\pi/4$	225°
1 (+1V)	0 (-1V)	S_3	$\pi/4$	45°
1 (+1V)	1 (+1V)	S_4	$7\pi/4$	315°

QPSK Transmitter :-



Bipolar converter :-

The input binary sequence is first converted to a bipolar type of signal. This signal is called b(t). It represents binary '1' by +1V and binary '0' by -1V.

Demultiplexing :-

The demultiplexer divides b(t) into two separate bit streams of the odd numbered and even numbered bits. $b_e(t)$ represents even numbered sequence $b_o(t)$ represents odd numbered sequence.

Modulation of quadrature carriers :-

The bit stream $b_e(t)$ modulates carrier $\sqrt{P_s}\cos(2\pi f_c t)$ and $b_o(t)$ modulates $\sqrt{P_s}\sin(2\pi f_o t)$. These modulators are balanced modulator.

$$s_e(t) = b_e(t)\sqrt{P_s}\sin(2\pi f_c t)$$

$$s_o(t) = b_o(t)\sqrt{P_s}\cos(2\pi f_c t).$$

Thus $s_e(t)$ and $s_o(t)$ are basically BPSK signals and they are similar to. BPSK. The only difference is that $T = 2T_b$. The value of $b_e(t)$ and $b_o(t)$ will be +1V or -1V.

Adder :- The adder adds these two signals $b_e(t)$ and $b_o(t)$. The output of the adder is QPSK signal

$$s(t) = s_e(t) + s_o(t)$$

$$= b_e(t)\sqrt{P_s}\sin 2\pi f_c t + b_o(t)\sqrt{P_s}\cos(2\pi f_c t).$$

phasor diagram of QPSK signal.

$s(t) = -\sqrt{P_s}\cos(2\pi f_c t) - \sqrt{P_s}\sin(2\pi f_c t)$

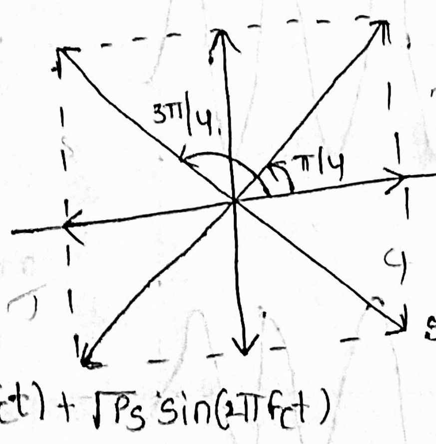
$b_o(t) = -1$

$b_e(t) = -1$

$s(t) = \sqrt{P_s}\cos(2\pi f_c t) - \sqrt{P_s}\sin(2\pi f_c t)$

$b_o(t) = 1$

$b_e(t) = -1$



$s(t) = \sqrt{P_s}\cos(2\pi f_c t) + \sqrt{P_s}\sin(2\pi f_c t)$

$b_o(t) = 1$

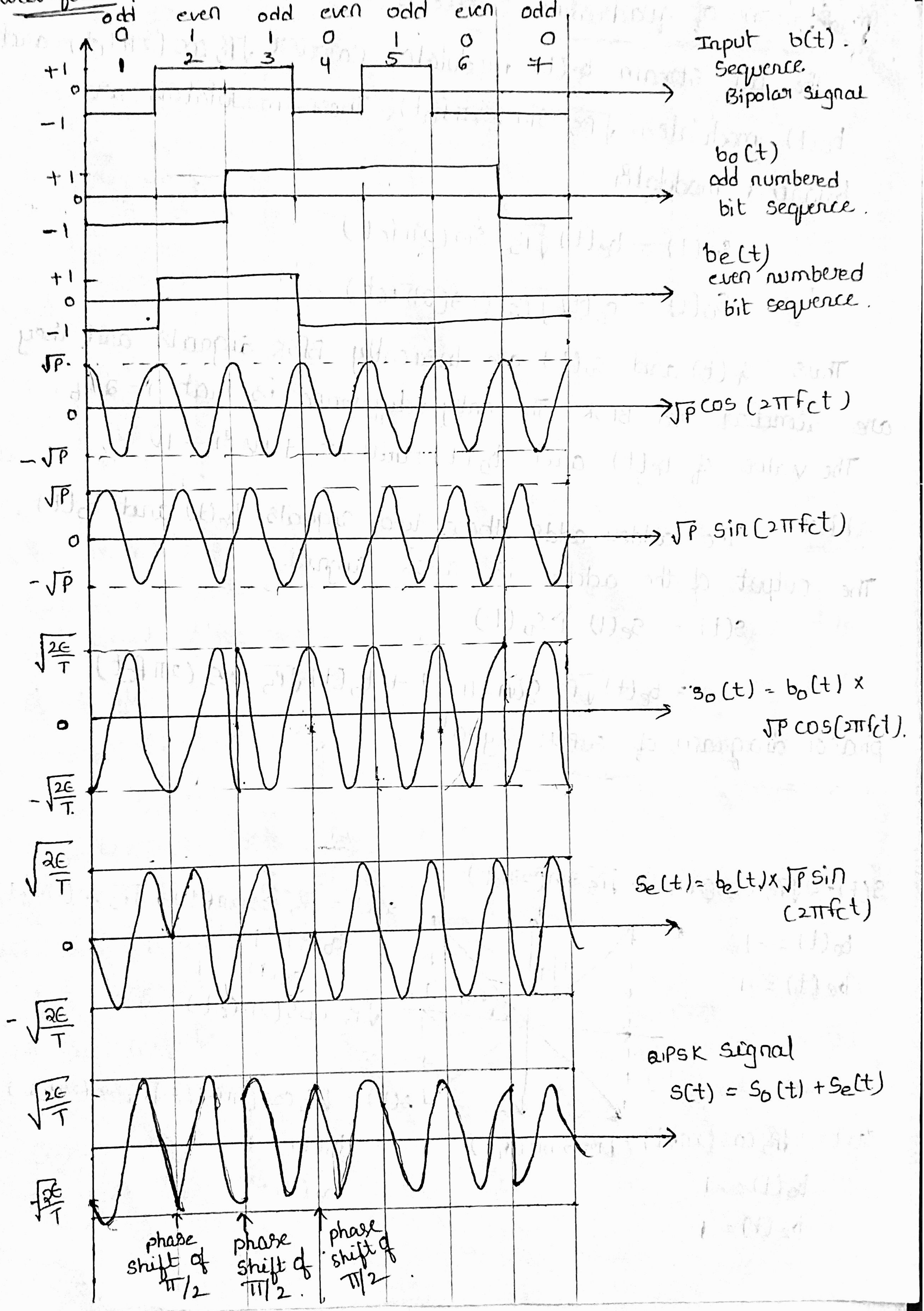
$b_e(t) = 1$

$s(t) = -\sqrt{P_s}\cos(2\pi f_c t) + \sqrt{P_s}\sin(2\pi f_c t)$

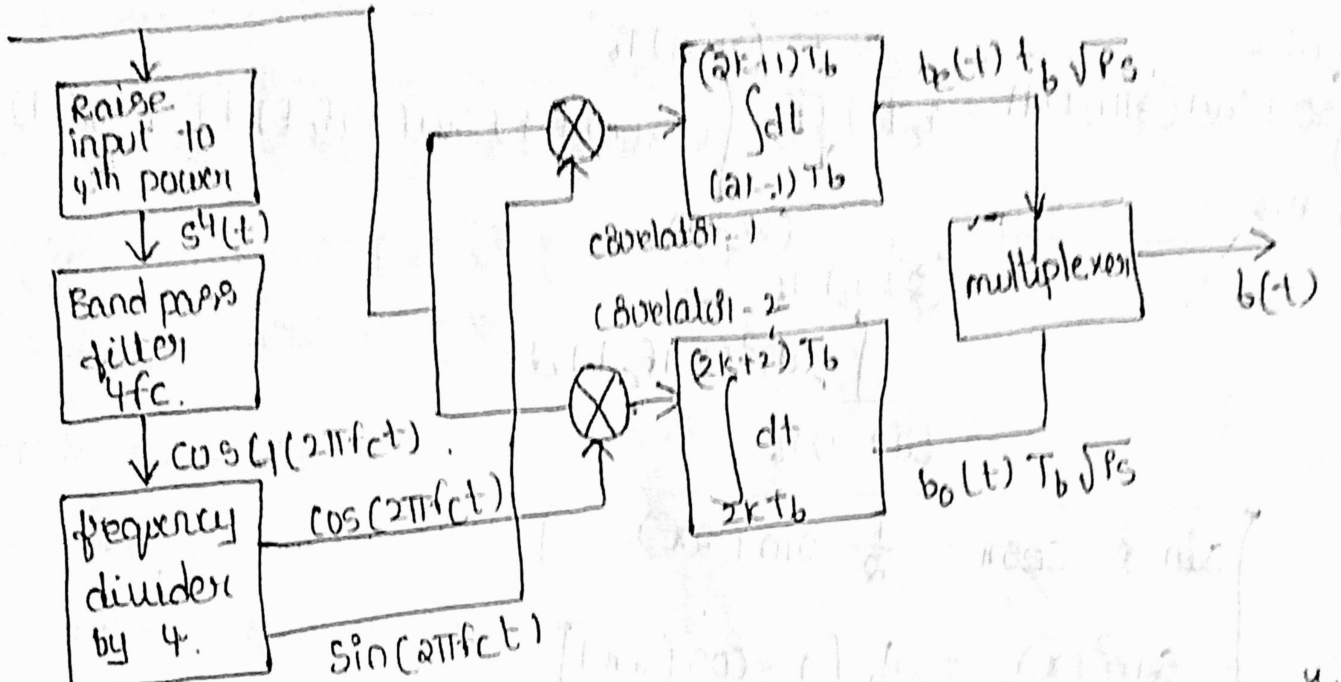
$b_o(t) = -1$

$b_e(t) = 1$

waveforms :-


odd
even
odd
even
odd
even
odd
0
1
1
0
1
0
0
1
2
3
4
5
6
7
+1
0
-1
Input b(t).
Sequence.
Bipolar signal
+1
0
-1
bo (t)
odd numbered
bit sequence.
+1
0
-1
be (t)
even numbered
bit sequence.
√P.
0
-√P
√P cos (2πfct)
√P
0
-√P
√P sin(2πfct)
√(2E/T)
0
-√(2E/T)
So (t) = bo (t) x
√P cos(2πfct).
√(2E/T)
0
-√(2E/T)
Se (t) = be (t) x √P sin
(2πfct)
QPSK signal
S(t) = So (t) + Se (t)
√(2E/T)
-√(2E/T)
phase shift of π/2
phase shift of π/2.
phase shift of π/2

QPSK Receiver:-



→ The received signal s(t) is first raised to 4th power i.e $s^4(t)$. Then it is passed through a band pass filter centered around 4fc. The output of the bandpass filter is a coherent carrier of frequency 4fc. This is divided by 4 and it gives two coherent carriers cos $(2\pi f_c t)$ and sin $(2\pi f_c t)$.

→ The coherent carriers are applied to two synchronous demodulators. These synchronous demodulators consists of multiplier and an integrator.

→ The integrator integrates the product signal over two bit intervals. ($T_s = 2T_b$)

→ The outputs of the two integrators are sampled at the one bit period T_b. Hence the output of multiplexer is the signal b(t). That is, the odd and even sequences are combined by multiplexer.

$$s(t)\sin(2\pi f_c t) = b_o(t)\sqrt{P_s}\cos(2\pi f_c t)\sin(2\pi f_c t) + b_e(t)\sqrt{P_s}\sin^2(2\pi f_c t)$$

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)dt = b_o(t)\sqrt{P_s}\int_{(2K-1)T_b}^{(2K+1)T_b}\cos(2\pi f_c t)\sin(2\pi f_c t)\,dt + b_e(t)\sqrt{P_s}\int_{(2k-1)T_b}^{(2k+1)T_b}\sin^2(2\pi f_c t)\,dt.$$

$$\boxed{\begin{aligned}\sin x \cdot \cos x &= \frac{1}{2}\sin(2x)\\ \sin^2(x) &= \frac{1}{2}\left[1-\cos(2x)\right]\end{aligned}}$$

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)dt = \frac{b_o(t)\sqrt{P_s}}{2}\int_{(2K-1)T_b}^{(2K+1)T_b}\sin(4\pi f_c t)dt + \frac{b_e(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}1.dt.$$

$$-\frac{b_e(t)\sqrt{P_s}}{2}\int_{(2K-1)T_b}^{(2K+1)T_b}\cos 4\pi f_c t\,dt.$$

In above equations, the first and third terms involves integration of sinusoidal carriers over two bit period. They have full cycles over two bit period and hence integration will be zero.

$$\int_{(2K-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)dt = \frac{b_e(t)\sqrt{P_s}}{2}\left[t\right]_{(2k-1)T_b}^{(2K+1)T_b}$$

$$= \frac{b_e(t)\sqrt{P_s}}{2} \times 2T_b$$

$$= b_e(t)\sqrt{P_s}\cdot T_b.$$

similarly the output of lower integrator as $b_o(t) . \sqrt{P_S}\, T_b$.

Signal space Representation of QPSK signals :-

In QPSK depending upon the combination of two successive bits, the phase shift occurs in carrier.

$$s(t) = \sqrt{2P} \cos(2\pi f_c t + \theta] \quad [\theta = \frac{\pi}{4}, (or)\ \frac{3\pi}{4} (or)\ \frac{5\pi}{4}\ or\ \frac{7\pi}{4})$$

it can be written as

$$s(t) = \sqrt{2P} \cos(2\pi f_c t + (2m+1)\frac{\pi}{4}] \quad \text{here } (m = 0, 1, 2, 3)$$

$$\cos[A+B] = \cos A . \cos B - \sin A \sin B.$$

The above equation expanded as

$$s(t) = \sqrt{2P} \cos(2\pi f_c t) \cos((2m+1)\frac{\pi}{4}] - \sqrt{2P} \sin(2\pi f_c t) \sin((2m+1)\frac{\pi}{4}]$$

$$s(t) = \sqrt{PT_S} \sqrt{\frac{2}{T_S}} \cos(2\pi f_c t) . \cos(2m+1)\frac{\pi}{4}] - \sqrt{PT_S} \sqrt{\frac{2}{T_S}} \sin(2\pi f_c t) \sin((2m+1)\frac{\pi}{4}]$$

$$\phi_1(t) = \sqrt{\frac{2}{T_S}} \cos(2\pi f_c t), \quad \phi_2(t) = \sqrt{\frac{2}{T_S}} \sin(2\pi f_c t).$$

$$s(t) = \sqrt{PT_S}\, \phi_1(t) \cos[(2m+1)\frac{\pi}{4}] - \sqrt{PT_S}\, \phi_2(t) \sin((2m+1)\frac{\pi}{4}]$$

$$s(t) = \sqrt{PT_S} \times \frac{\sqrt{2}}{\sqrt{2}} \cos[(2m+1)\frac{\pi}{4}] \phi_1(t) - \sqrt{PT_S} \times \frac{\sqrt{2}}{\sqrt{2}} \sin[(2m+1)\frac{\pi}{4}] \phi_2(t)$$

$$s(t) = \sqrt{P\frac{T_S}{2}} . \underbrace{\sqrt{2} \cos([2m+1)\frac{\pi}{4}]}_{b_o(t)} \phi_1(t) - \underbrace{\sqrt{P . \frac{T_S}{\sqrt{2}}} . \sqrt{2} \sin[(2m+1)\frac{\pi}{4}]}_{b_e(t)} \phi_2(t)$$

$$b_o(t) = \sqrt{2} \cos([2m+1)\frac{\pi}{4}]$$

$$b_e(t) = -\sqrt{2} \sin[(2m+1)\frac{\pi}{4}]$$

T_S = symbol duration

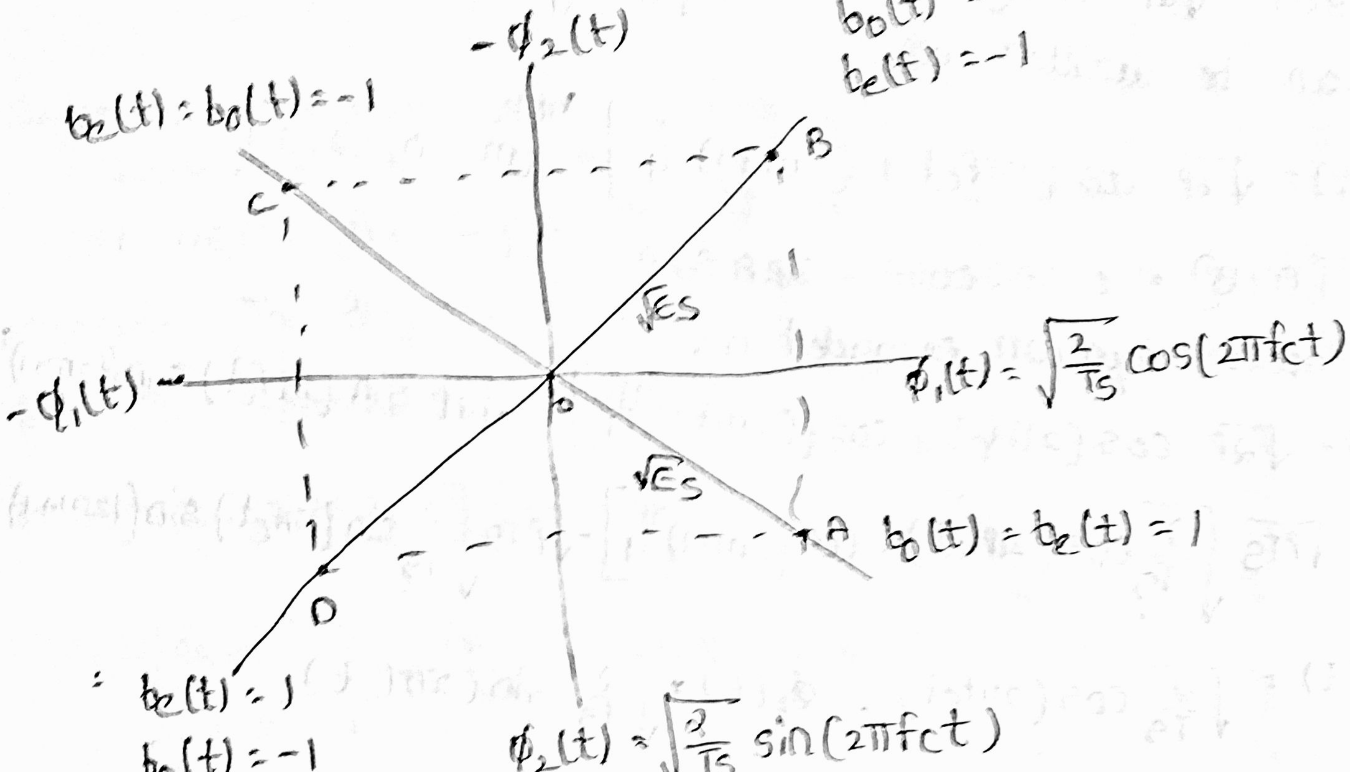
$T_S = 2T_b.$

$T_b = \frac{T_S}{2}$

$$s(t) = \sqrt{P_s T_b}\; b_o(t)\, \phi_1(t) + \sqrt{P_s T_b}\; b_e(t)\, \phi_2(t)$$

$$E_b = P_s \cdot T_b$$

$$\boxed{s(t) = \sqrt{E_b}\; b_o(t)\, \phi_1(t) + \sqrt{E_b}\; b_e(t)\, \phi_2(t)}$$



The distance of any signal point from o is given as.

distance o to B. $= \sqrt{P_s T_b + P_s T_b}$

$= \sqrt{2 P_s T_b} = \sqrt{\cancel{2} P_s \frac{T_s}{\cancel{2}}}$

$= \sqrt{P_s T_s}$

$= \sqrt{E_s}$.

Distance between signal points:-

for example signal points 'A' and B are two nearest points since they differ by.

$d^2 = (OA)^2 + (OB)^2$

$= \sqrt{E_s}^{\cancel{2}} + \sqrt{E_s}^{\cancel{2}}$

$= \sqrt{2E_s} = \sqrt{2.P_s.T_s} = \sqrt{2.P_s.2T_b}$

$$= \sqrt{2^2 . P_S . T_b}$$

$$= \sqrt{2^2} . \sqrt{P_S . T_b}$$

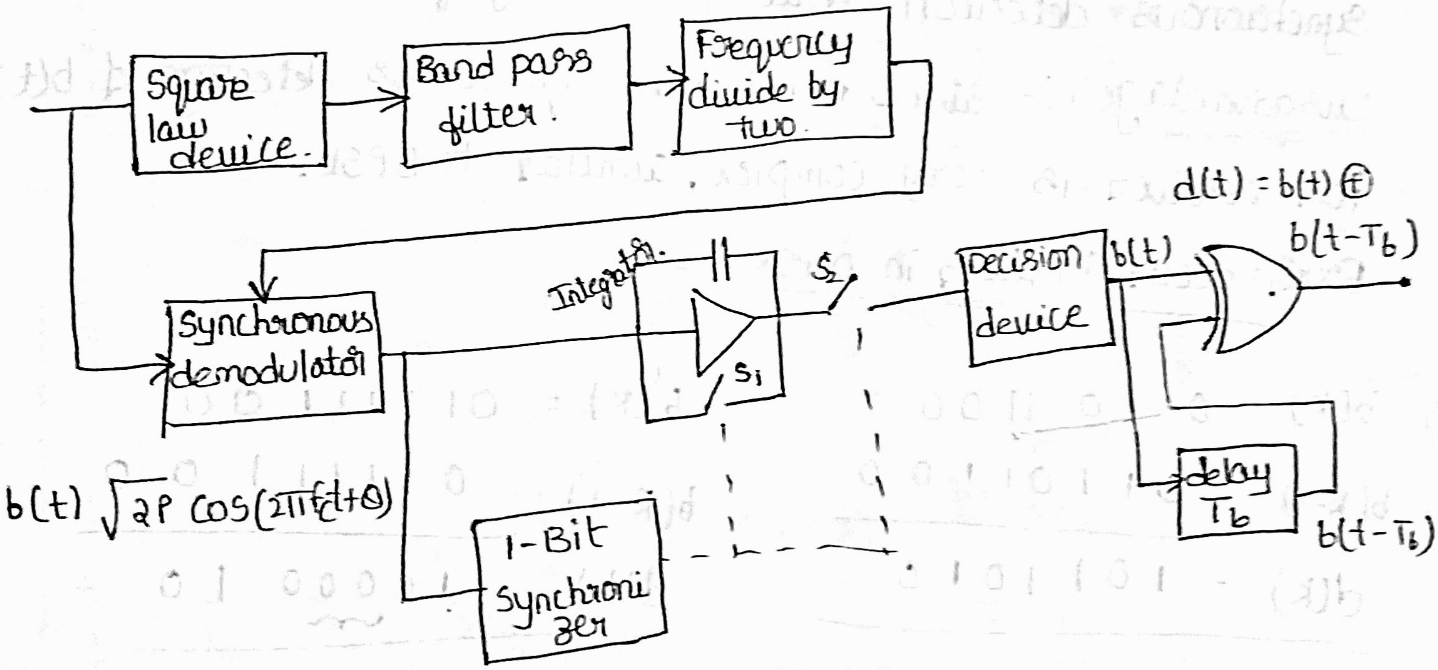
$$= 2 \sqrt{P_S . T_b}$$

$$= 2\sqrt{E_b} .$$

Differentially Encoded PSK :- (DEPSK).

In DEPSK, the signal b(t) is recovered using coherent detection. The original sequence d(t) is obtained by decoding b(t). The transmitter of DEPSK system is exactly similar to DPSK.

Receiver of DEPSK signal :-



Receiver block diagram of DEPSK system.

The receiver of DEPSK is synchronous or coherent detector. The signal b(t) $\sqrt{2P} \cos(2\pi f_c t + \theta)$ is received at the receiver. It is applied to the square law device and synchronous demodulator. The square law device, band pass filter and frequency divider detect the coherent carrier signal. The recovered carrier is given to synchronous demodulator. The output of demodulator is given to integrator and bit synchronizer. The output of the integrator is sampled at times $t = KT_b$. This sampled signal $S(KT_b)$ is given to the decision device. The output of decision device is the sequence $b(t)$. $b(t)$ is given to one input of the EX-OR gate and its delayed version of $b(t-T_b)$ is given to other input. The output of EX-OR gate is the sequence $d(t)$.

Advantages and disadvantages :-

Advantages :- The main advantage of DEPSK is that it uses synchronous detection. Hence probability of error is reduced.

Disadvantages :- Since DEPSK uses synchronous detection of $b(t)$ its receiver is very complex, similar to BPSK.

- Errors occur in pairs in DEPSK :-

$b(K)$ 0110 1100
$b(K-1)$ 01101100
$d(K)$ = 1011010

error free output.

$b'(K)$ = 01111100
$b(K-1)$ = 01111100
$d(K)$ 10000 10

one error is created in $b(t)$.

In DPSK there is a tendency of occuring errors in pairs. But single error can also occur in DPSK. But in DEPSK, errors always occur in pairs. This is because in DEPSK we make decision in each bit interval about the value of $b(t)$.

difference between modulation techniques —

SL. No	Parameter	ASK	PSK	FSK	DPSK	QPSK.
1.	modulation of	Amplitude	phase	frequency	phase	phase.
2.	Bits per symbol	one	one	one	one	Two.
3	Number of possible Symbols $m = 2^N$	Two	Two	Two	Two	Four.
4.	minimum distance.	$\sqrt{E_b}$	$2\sqrt{E_b}$	$\sqrt{2E_b}$	-	$2\sqrt{E_b}$
5	Band width.	$2f_b$	$2f_b$	$4f_b$	f_b	f_b.
6.	symbol duration.	T_b	T_b	T_b	$2T_b$	$2T_b$.
7.	Detection method.	coherent	coherent	non-coherent	non-coherent.	coherent
8.	Equation of the transmitted signal s(t)	$s(t) = \sqrt{2P}$ $\cos(2\pi f_c t)$.	$s(t) = b(t)$ $\sqrt{2P}$ $\cos(2\pi f_c t)$	$s(t) = \sqrt{2P}$ $\cos$ $(2\pi f_c + d(t)\Omega)t]$ Ω is frequency shift.	$s(t) = b(t)$ $\sqrt{2P}\cos$ $[2\pi f_c t)$.	$s(t) = \sqrt{2P}$ $\cos[2\pi f_c t +$ $(2m+1)\frac{\pi}{4}]$ $m = 0,1,2,3$.

M-Ary digital modulation Techniques :-

The basic digital modulation techniques which involve transmitting information in two levels. Hence they also be termed as binary digital modulation . techniques. we can extend the same principles to transmit information in more than two levels, in general m. levels. These modulation techniques is increased transmission rate on the same channel bandwidth. The signals with M different levels generated by changing the amplitude, frequency or phase of a carrier.

→ The main merit of m-ary techniques is increased transmission rate on the same channel bandwidth.

M-Array ASK :-

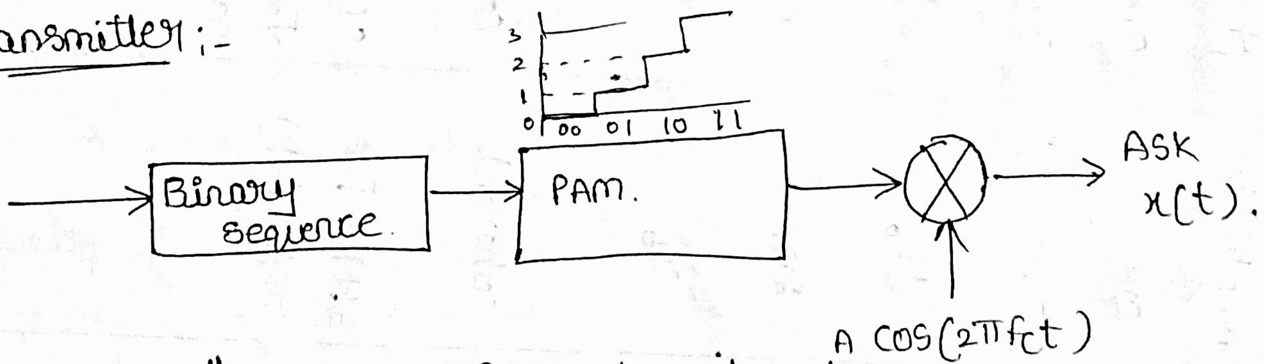
In m-Array the amplitude have M values. In this.

$M = 2^N$ (N = number of bites).

In this $N = \log_2 (M)$ bits.

Exp $M = 4$, $\log_2 (4) = 2$ bits.

Transmitter :-



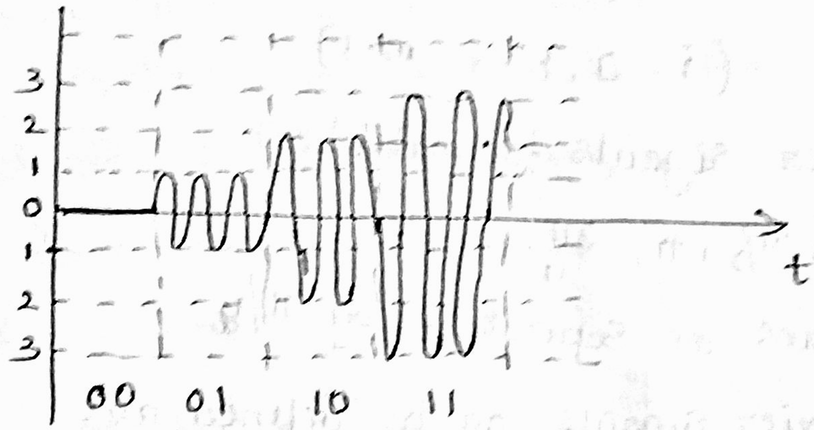
PAM is the transmission of data by varying the amplitude of the individual pulses in a regularly time sequence. If consider 4-Array. then. no.of bites is. 2 bits. Then Amplitude levels are 4.

They are $A_1 = 0 : 00$

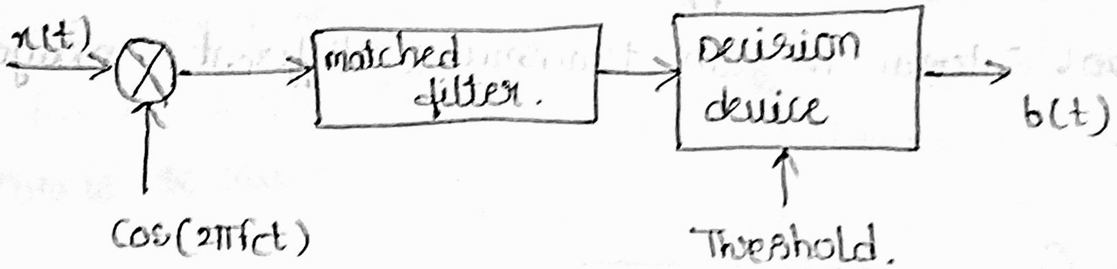
$A_2 = 1 \quad 01$

$A_3 = 2 \quad 10$

$A_4 = 3 \quad 11$



Receiver :-



The received signal x(t) is first multiplied to the carrier signal and then given to the matched filter. The matched filter is the optimal linear filter for maximizing the signal to noise ratio in the presence of additive noise. The decision device takes the decision at the end of every bit period. It compares the output of Filter with the threshold. Decision is taken in favour of '1' when threshold is exceeded. Decision is taken as '0' if threshold is not exceeded.

M - ary PSK :-

In BPSK, the phase of the carrier can taken on only two values 0° or 180°. In M-ary PSK can taken on M-different phase shift values within 2π range.

$$\phi = \frac{2\pi i}{M} \quad (i = 0, 1, \ldots, M-1)$$

Accordingly, M carrier signals for modulation.

when $M = 4$, $\phi_i = 0, \pi/2, \pi, \frac{3\pi}{4}$.

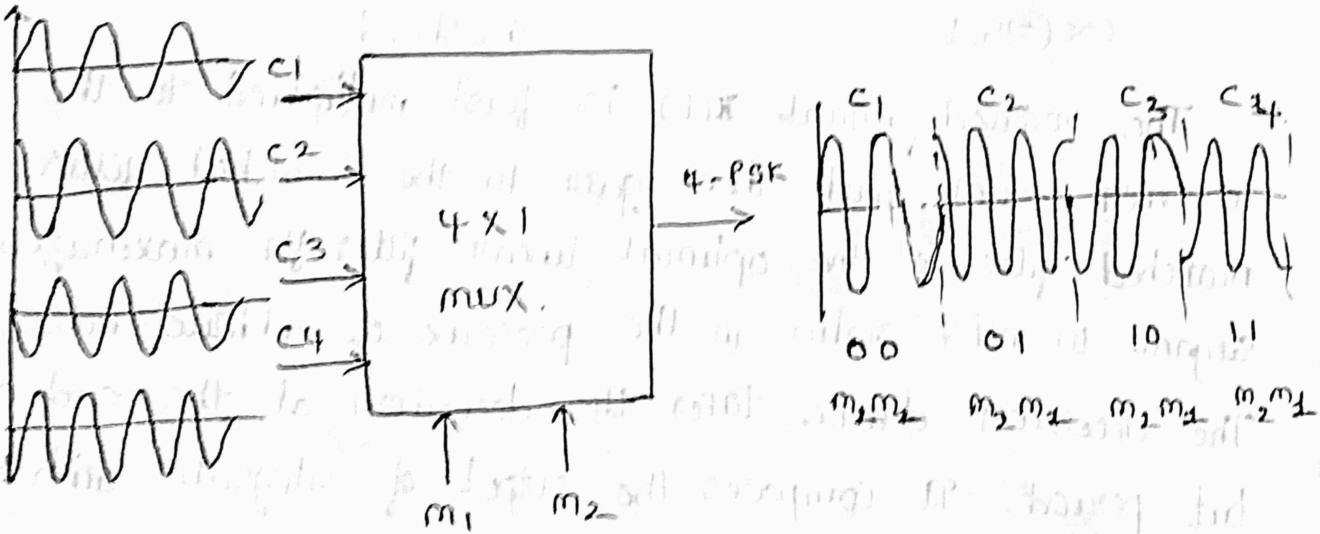
since the phase values are separated by $\pi/2$.

The M different carrier signals can be defined as:

$$C = A\cos\left(2\pi f_c t + \frac{2\pi i}{M}\right).$$

In a symbol interval we can transmit N different messages.

4 - Ary PSK :-

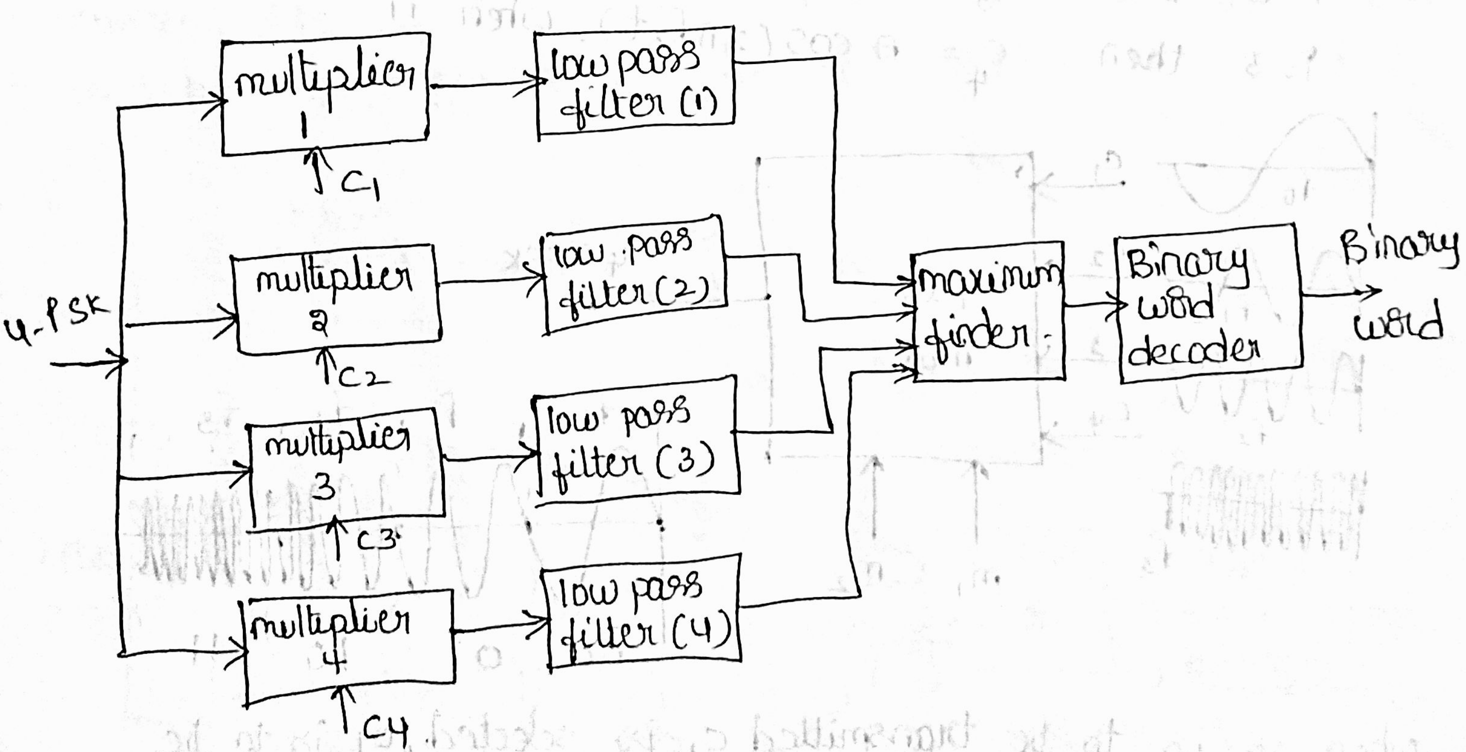


In a symbol interval we can transmit 2 different messages namely m_1, m_2 using the carriers c_1, c_2, c_3 and c_4 separated by π/2. For instance, 00 can be transmitted using a carrier with phase shift $\phi = 0°$, 01 with $\phi = 90°$, 10 with $\phi = 180°$ and 11 with $\phi = 270°$.

The two input message sequences are applied to the control inputs. when 00 is to be transmitted c_1 is selected, c_2 for 01, c_3 for 10 and c_4 for 11

Demodulation of 4-Ary PSK Signal:-

For the demodulation. only coherent detection is possible, In cohrent detection, incoming PSK signal is multiplied with four carrier signals c_1, c_2, c_3 and c_4. In given symbol interval. the multiplier whose carrier phase matches with that of the PSK signal will produce maximum output compared to other multipliers. Accordingly, the corresponding binary word of two bits is decoded. when multiplier 1 produces maximum output, then 00 is decoded. The two bit sequences can be separated to get the two messages m_1, and m_2.



The purpose of maximum finder is to find the channel that provides maximum output. Accordingly the binary word decoder will produce the corresponding binary word.

M-ary FSK :-

m-ary FSK is same as m-ary psk, except that the carriers are separated in frequency In m-ary fsk can take on m-different frequency values ginve by f_i where $i = 0, 1, \dots m-1$

The m-different carrier signals can be defined as

$$C = A\cos(2\pi f_i t) \qquad i = 0, 1, \dots m-1$$

4-FSK :- $m = 4$, 2^2

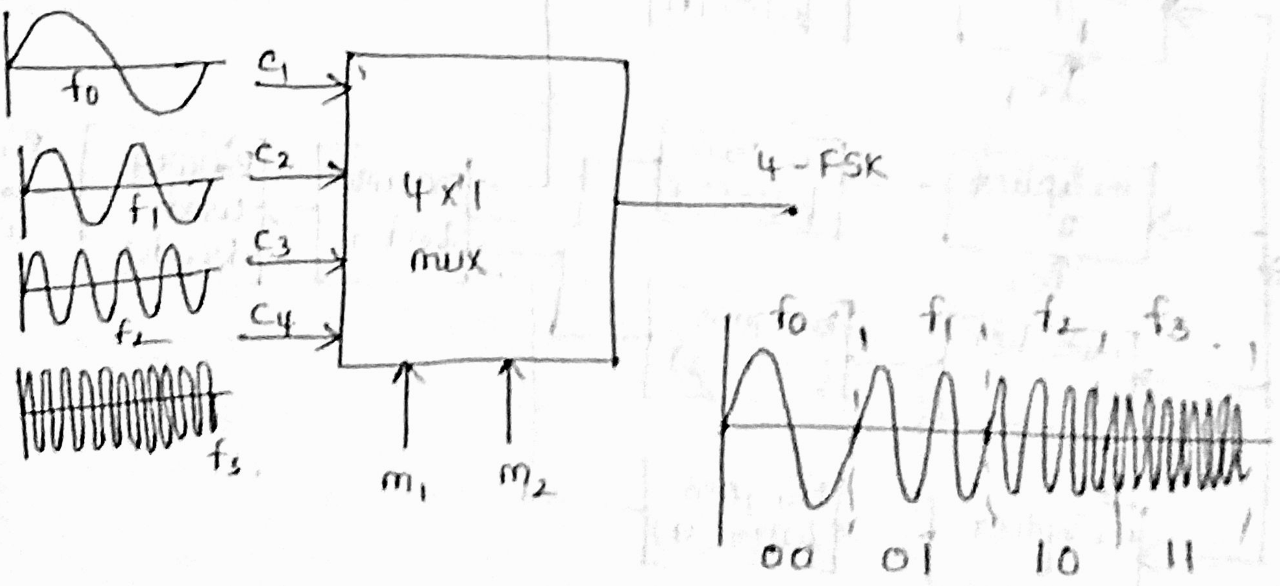
In a symbol interval we can transmit 2 different messages namely m_1, m_2 using the carriers c_1, c_2, c_3, c_4. For instance, 00 can be transmitted using a carrier with frequency f_0, 01 with f_1, 10 with f_2 and 11 with f_3.

$i = 0$, then $C_1 = A\cos(2\pi f_0 t)$ when 00

$i = 1$, then $C_2 = A\cos(2\pi f_1 t)$ when 01

$i = 2$, then $C_3 = A\cos(2\pi f_2 t)$ when 10

$i = 3$ then $C_4 = A\cos(2\pi f_3 t)$. when 11

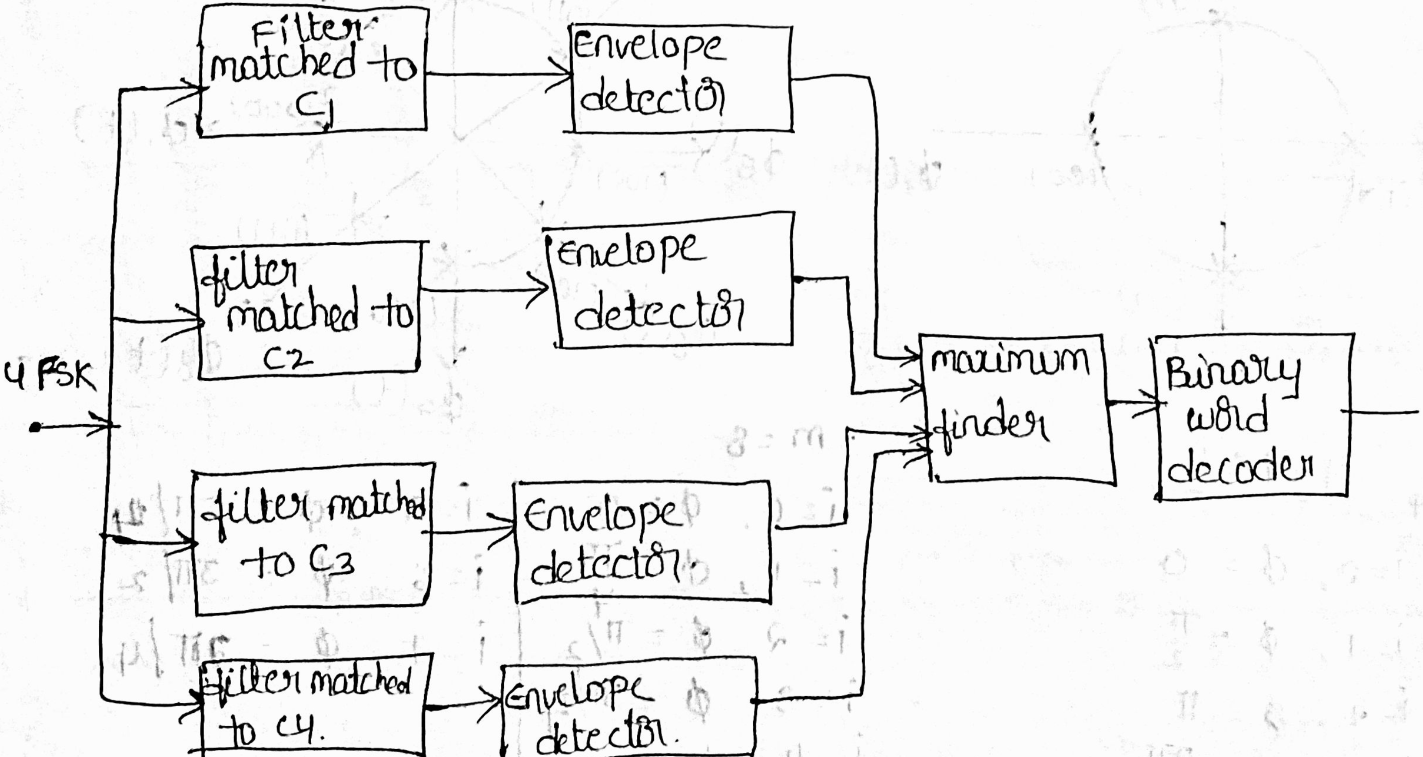


when 00 is to be transmitted c_1 is selected, 01 is to be transmitted c_2 is selected ; 10 is to be transmitted c_3 is selected, 11 is to be transmitted c_4 is selected.

Demodulation m-ary FSK signal :-

In coherent detection incoming FSK signal is applied to four analog multipliers having carrier signals $c_1, c_2, c_3,$ and c_4 which are separated in frequency. In a given symbol interval, the analog multiplier whose carrier frequency matches with that of the FSK signal will produce maximum output. Accordingly the corresponding binary word of two bits is decoded. The two bit sequences can be separated to get the two messages m_1 & m_2.

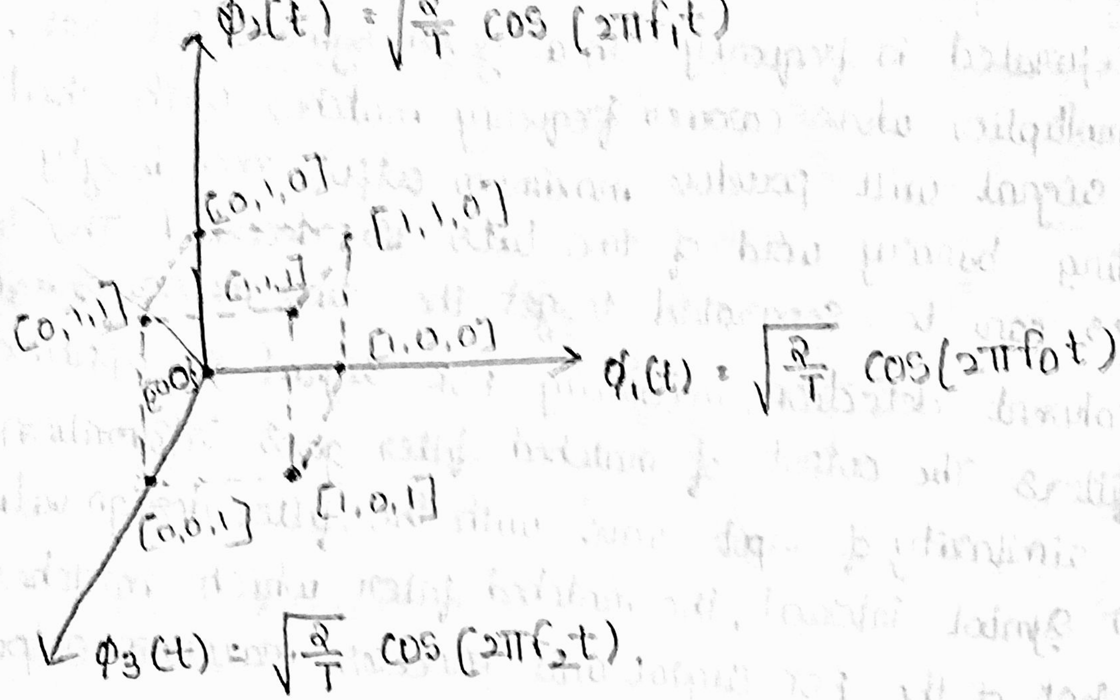
In non-coherent detection, incoming FSK signal is applied to four filters. The output of matched filter gives information about the similarity of input wave with the filter design value. In a given symbol interval, the matched filter which matches best with that of the FSK signal will produces maximum output compared to other filters. The output of the matched filters are passed through the envelope detectors. The output of the envelope detectors are compared and one with maximum output is taken as the channel and its corresponding binary word is decoded.



Non-coherent detection of 4-FSK signal.

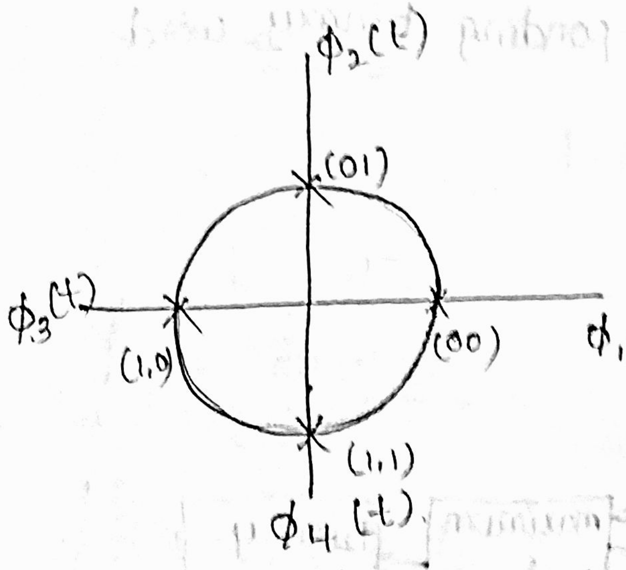
Signal space representation :- (constillation diagram).

8-FSK :-

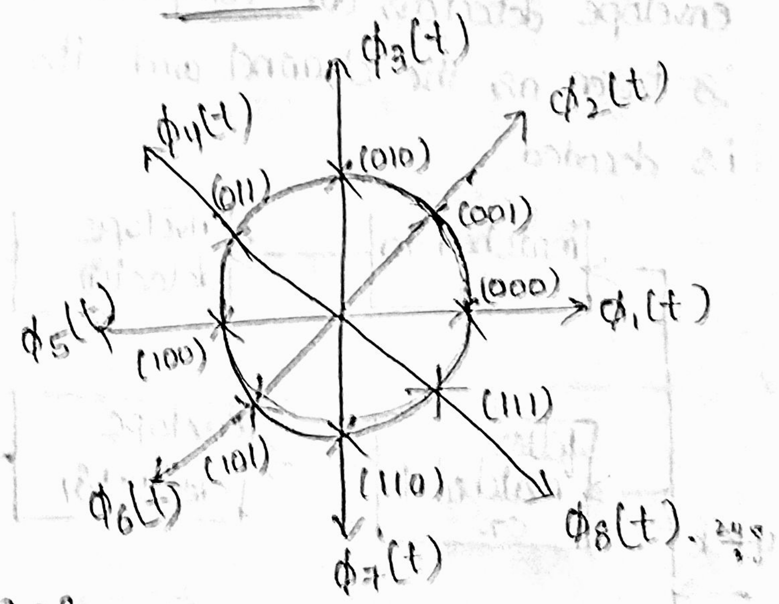


16-FSK becomes imaginary.

4-PSK :-



8-PSK



4-PSK

M=4

i=0, $\phi = 0$

i=1, $\phi = \frac{\pi}{2}$

i=2, $\phi = \pi$

i=3, $\phi = \frac{3\pi}{2}$

M=8

i=0, $\phi = 0$

i=1, $\phi = \frac{\pi}{4}$

i=2 $\phi = \pi/2$

i=3 $\phi = 3\pi/4$

i=4 $\phi = \pi$

i=5, $\phi = 5\pi/4$

i=6 $\phi = 3\pi/2$

i=7 $\phi = 7\pi/4$.

DATA TRANSMISSION

Introduction :-

A significant portion of communication, these days is in analog form, it is now being replaced rapidly by digital communication. The data consists of only two symbols '1' and '0'. The resulting sequence of these pulses in transmitted over a channel. At the receiving end, these pulses are detected and are converted back to binary data.

→ This means that a signal received at a particular time is due to the transmitted pulse and some interference due to the adjacent pulses. This interference due to adjacent pulses is known as Inter symbol Interference (ISI).

→ Any Transmission channel adds noise to the signal. This noise is known as the channel noise.

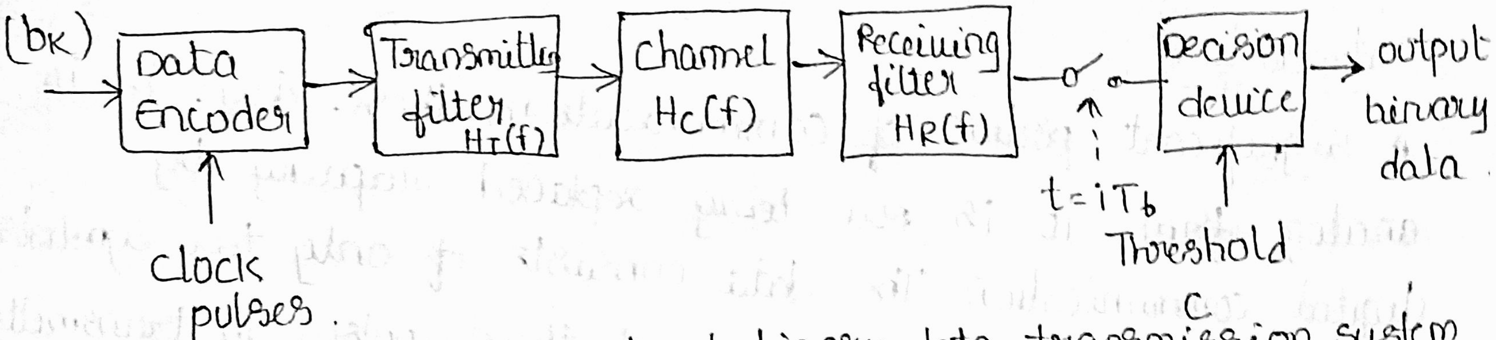
→ This means that the ISI and channel noise simultaneously affect the received signal and create errors in the signal.

→ Special cares must be taken to reduce the noise and increase the signal to noise ratio.

Base band Transmission of Binary data :-

→ when the signal is transmitted over the channel, without any modulation, it is called base band transmission.

→ one of the major problem occured in base band transmission is intersymbol interference. This interference takes place due to nature of the channel.



Block diagram of baseband binary data transmission system

→ In baseband transmission, there is no modulation of high frequency carrier. One of the base band system for transmission of digital data is discrete pulse amplitude modulation. (PAM)

→ The discrete PAM can have only two amplitude levels. ('1' or '0'). Successive binary digits can be combined into symbols. There can be multiple amplitude levels corresponding to these symbols.

→ The binary data bK is applied to the data encoder. The data encoder generates the pulse waveform x(t). This waveform can be represented as.

$$x(t) = \sum_{K=-\infty}^{\infty} A_K\, g(t - KT_b)$$

Here T_b is the duration of each input binary bit.

g(t) is the shaping pulse.

and $A_K = \begin{cases} +a & \text{if } bK = 1 \\ -a & \text{if } bK = 0 \end{cases}$

→ The signal x(t) is then passed through the transmitting filter. The combined transfer function of the transmitting filter is $H_T(f)$. The signal is then passed through the channel having the transfer function $H_C'(f)$.

→ The channel delivers the signal to the receiving filter. It consists of all the necessary receiving circuits and systems and the transfer function of the receiving filter is $H_R(f)$.

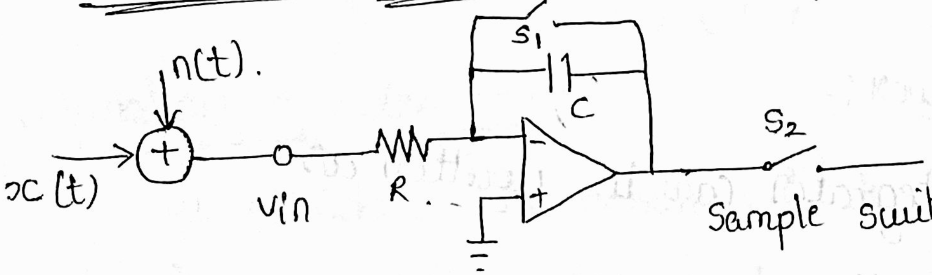
→ The output of the receiving filter is $y(t)$. The signal $y(t)$ is sampled synchronously with the transmitter. The sampling instants are $t = iT_b$. The sampled signal $y(t_i)$ is then given to the decision device.

→ The decision device compares the input signal with threshold 'c'. Then the decision is taken.

If $y(ti) > c$ then symbol '1'.

If $y(ti) \leq c$ the symbol '0'.

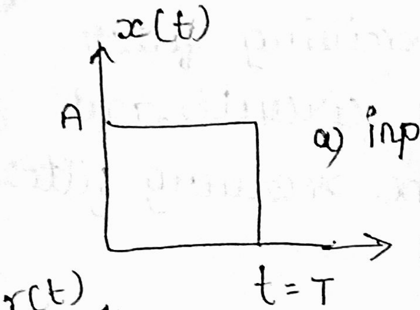
Base band signal Receiver :- [integrate and dump filter]



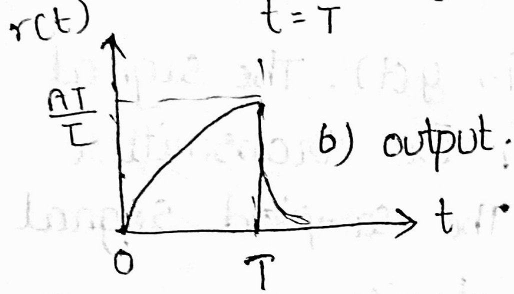
integrate and dump filter

→ The digital signal $x(t)$ is corrupted by noise $n(t)$ during Transmission over channel. such noisy signal $[x(t) + n(t)]$ is given to the input of integrate and Dump filter. The capacitor is discharged fully at the begining of the bit interval.

→ The integrator then integrate noisy input signal over one bit period. For the square pulse input, the output of the integrator will be a triangular pulse.



a) input



b) output.

→ At the end of bit period t = T, the value of r(t) reaches to its maximum amplitude. Therefore the value of r(t) is sampled at the end of bit period.

→ Depending upon the value of r(T), the decision is taken. Thus integrator integrates independent of the value of previous bit. The output of integrator will decrease after t > T.

Signal to Noise Ratio of the integrator and Dump filter :-

$$\frac{S}{N} = \frac{\text{Signal power}}{\text{Noise power}}$$

Calculation of signal power :-

The output of the integrator can be written as.

$$r(t) = \frac{1}{RC}\int_0^T v_{in}\,dt$$

$$= \frac{1}{RC}\int_0^T [x(t) + n(t)]\,dt$$

$$= \frac{1}{RC}\int_0^T x(t)\,dt + \frac{1}{RC}\int_0^T n(t)\,dt.$$

$$= x_0(t) + n_0(t).$$

Here $x_0(t)$ is the output signal voltage and $n_0(t)$ is the output noise. Consider output signal ~~power~~ voltage.

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\,dt$$

The value of $x(t) = A$ from 0 to T.

$$x_0(t) = \frac{1}{RC}\int_0^T A\cdot dt$$

$$= \frac{A}{RC}\int_0^T 1\cdot dt$$

$$= \frac{A}{RC}\left[t\right]_0^T$$

$$= \frac{At}{RC} \quad [\text{Time constant } RC = \tau]$$

$$x_0(t) = \frac{AT}{\tau}$$

output signal power $= \dfrac{x_0^2(t)}{1\Omega} = x_0^2(t)$

output signal power $: \left(\dfrac{AT}{\tau}\right)^2 = \dfrac{A^2T^2}{\tau^2}$

Calculation of Noise power :-

In standard 1Ω resistance, the noise power will be.

$$\frac{\overline{n_0^2(t)}}{1\Omega} = n_0^2(t).$$

Here mean square value of noise is taken since it is random signal.

Noise power $\overline{N_0^2(t)} = \int_{-\infty}^{\infty} S_{no}(f)\,df.$

→ The input and output power spectral densities are.

$$S_{no}(f) = |H(f)|^2 S_{ni}(f).$$

H (f) is transfer function of filter.

$S_{no}(f)$ is psd of output noise and

$S_{ni}(f)$ is psd of input noise.

→ we are assuming that white noise is present. The power spectral density of this noise is

$$S_{ni}(f) = \frac{N_0}{2}.$$

then $$S_{no}(f) = |H(f)|^2 . \frac{N_0}{2}.$$

→ A network which performs integration operation has the transfer function is

$$H(f) = \frac{\frac{1}{CS}}{R} = \frac{1}{CSR}$$

$$= \frac{1}{j\omega RC}$$

→ A delay of t = T in time domain is equivalent to $e^{-j\omega T}$ in frequency domain.

then $$H(f) = \frac{1-e^{-j\omega T}}{j\omega RC}.$$

$\omega = 2\pi f$, $RC = \tau$.

$$H(f) = \frac{1-e^{-j2\pi fT}}{j2\pi f\tau} \quad [\because e^{-j2\pi f\tau} = \cos(2\pi fT) - j\sin(2\pi fT)]$$

$$H(f) = \frac{1-[\cos(2\pi fT) - j\sin(2\pi fT)]}{j2\pi f\tau}.$$

Separate the real and imaginary parts.

$$H(f) = \frac{\sin 2\pi fT}{2\pi f\tau} + \frac{1-\cos(2\pi fT)}{j\ 2\pi f\tau}$$

$a+jb =$

$|a+jb| = \sqrt{a^2+b^2}$

$|a-jb| = \sqrt{a^2+b^2}$

$$|H(f)| = \sqrt{\frac{\sin^2(2\pi f T) + (1-\cos(2\pi f T))^2}{(2\pi f \tau)^2}}$$

$$|H(f)|^2 = \sqrt{\frac{\sin^2(2\pi f T) + (1-\cos(2\pi f T))^2}{(2\pi f \tau)^2}}^{\cancel{2}}$$

$$\boxed{(a-b)^2 = a^2+b^2-2ab}$$

$$|H(f)|^2 = \frac{\sin^2(2\pi f T) + 1 + \cos^2(2\pi f T) - 2\cos(2\pi f T)}{(2\pi f \tau)^2}$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$|\cancel{\$}H(f)|^2 = \frac{2 - 2\cos(2\pi f T)}{(2\pi f T)^2}$$

$$\boxed{1 - \cos\theta = 2 . \sin^2\frac{\theta}{2}}$$

$$|H(f)|^2 = \frac{2(1-\cos(2\pi f T))}{4(\pi f \tau)^2} = \frac{\cancel{2}.\cancel{2}. \sin^2\frac{\cancel{2}\pi f T}{\cancel{2}}}{\cancel{4}(\pi f \tau)^2}$$

$$|H(f)|^2 = \frac{\sin^2(\pi f T)}{(\pi f \tau)^2}$$

then value of $\overline{n_o^2(t)}$ is given as.

$$\overline{n_o^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot \frac{N_o}{2}\, df.$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2(\pi f T)}{(\pi f \tau)^2} \cdot \frac{N_o}{2}\, df.$$

$$= \frac{N_o}{2}\int_{-\infty}^{\infty} \frac{\sin^2(\pi f T)}{(\pi f \tau)^2}\, df.$$

$$(\pi f \tau) = x$$

$$dx = \pi\tau\, df \quad a \quad : \quad df = \frac{1}{\pi\tau}\, dx.$$

$$f = \frac{x}{\pi\tau}$$

$$\pi f T = \frac{xT}{\tau}$$

then $$n_o^2(t) = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{xT}{\tau}\right)}{x^2}\cdot\frac{1}{\pi\tau}\,dx.$$

$$n_o^2(t) = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)^2}\cdot\left(\frac{T}{\tau}\right)^2\cdot\frac{1}{\pi\tau}\,dx$$

$$= \frac{N_0}{2}\cdot\frac{T^2}{\pi\tau^3}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)^2}\cdot dx.$$

let $\frac{xT}{\tau} = u$

$$dx = \frac{\tau}{T}\,du$$

then $$n_o^2(t) = \frac{N_0}{2}\cdot\frac{T^2}{\pi\tau^3}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}\cdot\frac{\tau}{T}\,du.$$

$$= \frac{N_0}{2}\cdot\frac{\cancel{T^2}}{\pi\cancel{\tau^3}}\cdot\frac{\cancel{\tau}}{\cancel{T}}\int_{-\infty}^{\infty}\left(\frac{\sin u}{u}\right)^2 du$$

Since the function can be written as

$$n_o^2(t) = \frac{N_0T}{2\pi\tau^2}\cdot 2\int_{0}^{\infty}\left(\frac{\sin u}{u}\right)^2 du. \qquad \left(\frac{\sin\theta}{\theta}\right)^2 = \frac{\pi}{2}$$

$$= \frac{N_0T}{2\cancel{\pi}\tau^2}\cdot\cancel{2}\cdot\frac{\cancel{\pi}}{\cancel{2}}$$

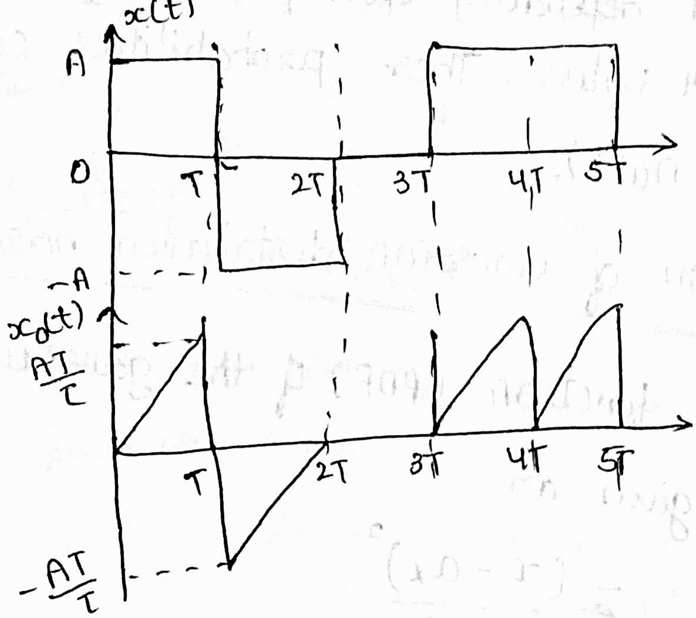
$$\boxed{n_o^2(t) = \frac{N_0T}{2\tau^2}}$$

Calculation of signal to noise ratio :-

$$\frac{S}{N} = \frac{\frac{A^2T^2}{\tau^2}}{\frac{N_0T}{2\tau^2}} = \frac{A^2T^{\cancel{2}}}{\cancel{\tau^2}}\times\frac{2\cancel{\tau^2}}{N_0\cancel{T}} = \frac{2A^2T}{N_0}$$

$$\boxed{\frac{S}{N} = \frac{A^2T}{N_0/2}}$$

→ The signal to noise ratio improves in proportion to sampling period 'T'. it also increases as signal amplitude 'A' is more.



input signal to the integrate and dump filter.

output signal of the integrate and dump filter.

probability of error in Integrate and Dump filter Receiver :-

→ probability of error P_e is the good measure for performance of the detector. The output of the integrator is given as.

$$r(t) = x_o(t) + n_o(t).$$

in this $x_o(t) = \frac{AT}{\tau}$ for $x(t) = A$.

$$x_o(t) = -\frac{AT}{\tau} \text{ for } x(t) = -A.$$

Therefore $r(t) = \frac{AT}{\tau} + n_o(t)$ for $x(t) = A$

$$r(t) = -\frac{AT}{\tau} + n_o(t) \text{ for } x(t) = -A.$$

→ consider that $x(t) = -A$ Then if noise $n_o(t) > \frac{AT}{\tau}$, output $r(t)$ will be positive according to the equation. Then the receiver will decide in favour of symbol $+A$, which is wrong decision. Thus error is introduced.

→ Similarly $x(t) = +A$, then if noise $n_o(t) < -\frac{AT}{\tau}$, output $r(t)$ will be negative according to equation. This leads to decision in favour of $-A$.

→ The error is introduced depending upon probability that $n_o(t)$ takes a particular value. These probabilities can be obtained from PDF of $n_o(t)$.

Probability density function of Gaussian distributed noise :-

The probability density function (PDF) of the gaussian distributed function is given as.

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \, e^{-\frac{(x - a_x)^2}{2\sigma_x^2}}$$

it can be written as

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x-m)^2/2\sigma^2}$$

Here $f_x(x)$ is the PDF of random function x.

m is the mean value and σ is the standard deviation.

then $x = n_o(t)$.

since this noise has zero mean value, $m = 0$.

$$f_x(n_o(t)) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-[n_o(t)]^2/2\sigma^2}$$

→ The standard deviation σ is given as.

$$\sigma^2 = [m_2 - (m_1)^2]$$

$$\sigma_x = [\overline{x^2} - m_x^2]$$

$$\overline{x}^2 = n_o^2(t) = \frac{N_oT}{2\tau^2}$$

$$\sigma = \sqrt{[n_0^2(t)]} = \sqrt{\frac{N_0 T}{2\tau^2}}$$

Hence equation $f_X(n_0(t)] = \frac{1}{\sqrt{\frac{N_0 T}{2\tau^2}}\sqrt{2\pi}} \cdot e^{-[n_0(t)]^2 / 2\left(\frac{N_0 T}{2\tau^2}\right)}$

on simplifying above equation we get

$$f_X[n_0(t)] = \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2 / \frac{N_0 T}{\tau^2}}$$

Thus equation gives PDF of white gaussian noise.

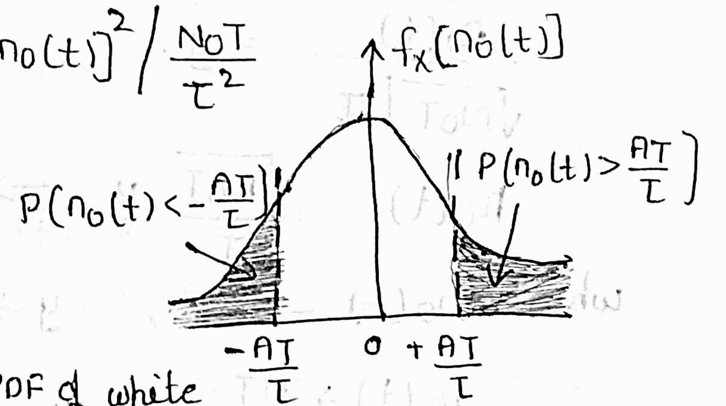


fig. PDF of white gaussian noise of zero mean.

Evaluation of error probability from error conditions :-

→ From the property of PDF is .

$$P\left(n_0(t) > \frac{AT}{\tau}\right) = \int_{\frac{AT}{\tau}}^{\infty} f_X(n_0(t)]\, d(n_0(t)]$$

This equation gives the probability that $n_0(t)$ takes value greater than $\frac{AT}{\tau}$.

→ Similarly the probability that $n_0(t)$ takes value less than $\frac{AT}{\tau}$ is given by area under the curve from $-\frac{AT}{\tau}$ onwards on left side.

$$P\left(n_0(t) > \frac{AT}{\tau}\right) = P\left[n_0(t) < -\frac{AT}{\tau}\right)$$

The probability of error is given by

$$P_e = P\left(n_0(t) > \frac{AT}{\tau}\right) = P\left[n_0(t) < -\frac{AT}{\tau}\right).$$

$$P_e = P\left(n_0(t) > \frac{AT}{\tau}\right) = \int_{\frac{AT}{\tau}}^{\infty} f_X(n_0(t).\, d(n_0(t)]$$

$$P_e = \int_{\frac{AT}{T}}^{\infty} \frac{T}{\sqrt{\pi N_0 T}} \, e^{-[n_0(t)]^2 / \left(\frac{N_0 T}{T^2}\right)} \, d[n_0(t)]$$

put $$\frac{[n_0(t)]^2}{\left[\frac{N_0 T}{T^2}\right]} = y^2$$

$$\frac{n_0(t)}{\sqrt{N_0 T}/T} = y.$$

$$n_0(t) = \frac{\sqrt{N_0 T}}{T} y. \Rightarrow d[n_0(t)] = \frac{\sqrt{N_0 T}}{T} dy.$$

when $n_0(t) \to \infty$, $y \to \infty$

$$n_0(t) \to \frac{AT}{T} \qquad y = \frac{AT/T}{\sqrt{N_0 T}/T} = \frac{AT}{\sqrt{N_0 T}} = \sqrt{\frac{A^2 T}{N_0}}$$

then $$P_e = \int_{\sqrt{\frac{A^2 T}{N_0}}}^{\infty} \frac{\cancel{T}}{\sqrt{\pi \cancel{N_0 T}}} e^{-y^2} \cdot \sqrt{\frac{\cancel{N_0 T}}{\cancel{T^2}}} \, dy$$

$$P_e = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{AT}{N_0}}}^{\infty} e^{-y^2} dy. \Rightarrow P_e = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \int_{\sqrt{\frac{A^2 T}{N_0}}}^{\infty} e^{-y^2} dy$$

→ The integration inside brackets can be evaluated with the help of complementary error function.

$$\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2 T}{N_0}}}^{\infty} e^{-y^2} dy = \text{erfc}(u).$$

→ This is a standard result and normally evaluated using numerical methods.

$$A^2 T = E.$$

$$P_e = \frac{1}{2} \text{erfc} \sqrt{\frac{A^2 T}{N_0}} \Rightarrow \boxed{P_e = \frac{1}{2} \text{erfc} \sqrt{\frac{E}{N_0}}}$$

Optimum Receiver :

Definition of optimum receiver :-

→ In the last section we stuided integrate and Dump receiver on that integrator is the optimum filter for the purpose of minimizing the probability of error? For this, we will study a generalized filter for receiving binary word (coded) Signals. It is called optimum filter.

Decision boundary :-

→ let's assume that the received signal is a binary waveform. is used to represent binary 1's and 0's

for binary '1' $x_1(t) = +A$ for one bit period 'T' and

for binary '0' $x_2(t) = -A$ for one bit period 'T'.

→ Thus the input signal $x(t)$ will be either $x_1(t)$ or $x_2(t)$ depending upon the polarity of the input signal.

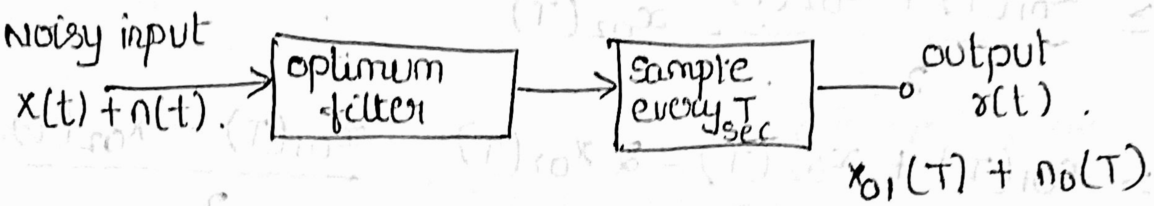


fig:- A Receiver for binary coded signal. (or)

$x_{02}(T) + n_0(T)$.

→ Noise $n(t)$ is added to the signal $x(t)$ over the channel during transmission. Hence input to the optimum filter is $x(t)+n(t)$. and output from the receiver is $x_{01}(T)+n_0(T)$ or $x_{02}(T)+n_0(T)$.

→ In the absence of noise $n(t)$, the output of the receiver will be.

$r(T) = x_{01}(T)$ if $x(t) = x_1(t)$ and

$r(T) = x_{02}(T)$ if $x(t) = x_2(t)$.

→ Thus in the absence of noise, decisions are taken clearly. But if noise is present then, Select $x_1(t)$ if $r(T)$ is closer to $x_{01}(T)$ then $x_{02}(T)$, and select $x_2(t)$ if $r(T)$ is closer to $x_{02}(T)$ then $x_{01}(T)$.

→ Decision boundary $= \frac{x_{01}(T) + x_{02}(T)}{2}$.

→ Probability of error of optimum filter:-

Error conditions:-

→ The probability of error can be obtained on the suppose that $x_2(t)$ was transmitted, but $x_{01}(T)$ is greater than $x_{02}(T)$. If $n_0(T)$ is positive and larger in magnitude then the voltage difference $\frac{1}{2}[x_{01}(T) + x_{02}(T)] - x_{02}(T)$, then incorect decision will be taken. Error will be generated if:

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T)$$

$$n_0(T) > \frac{x_{01}(T) + x_{02}(T) - 2x_{02}(T)}{2} = \frac{x_{01}(T) - x_{02}(T)}{2}.$$

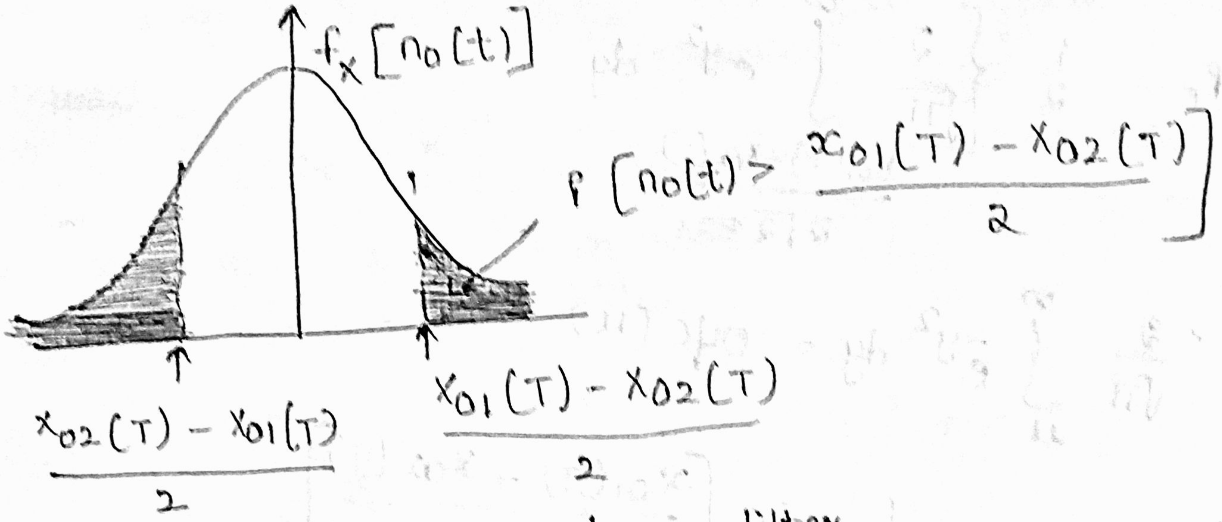
→ The probability density function (PDF) for $n_0(t)$ is given as.

$$f_x(n_0(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2}$$

Evaluation of error probability:-

$$P_e = P\left[n_0(t) > \frac{x_{01}(T) - x_{02}(T)}{2}\right] = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} f_x[n_0(t)] \cdot d[n_0(t)]$$

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(n_0(t))^2/2\sigma^2} d[n_0(t)]$$



calculation of P_e for optimum filter.

→ let $\frac{[n_o(t)]^2}{2\sigma^2} = y^2 \Rightarrow [n_o(t) = \sqrt{y^2 2\sigma^2}$

$$n_o(t) = \sigma\sqrt{2}\,y.$$

$$d[n_o(t)] = \sigma\sqrt{2}\,dy$$

when $n_o(t) = \infty \rightarrow y = \infty$

$n_o(t) = \frac{x_{01}(T)-x_{02}(T)}{2}$, $y = \frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}$

than $P_e = \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2} \sigma\sqrt{2}\,dy.$

then $P_e = \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}.\sqrt{\pi}} e^{-y^2}.\cancel{\sigma}\cancel{\sqrt{2}}\,dy$

$$P_e = \frac{1}{\sqrt{\pi}} \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} e^{-y^2}.\,dy.$$

$$P_e = \frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} e^{-y^2}\, dy\right.$$

$$\frac{2}{\sqrt{\pi}}\int_{u}^{\infty} e^{-y^2}\, dy = \operatorname{erfc}(u).$$

then $$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{x_{01}(T) - X_{02}(T)}{2\sqrt{2}\sigma}\right]$$

This is the required expression for error probability P_e of optimum filter. It may be noted that the "erfc" function is the monotonically decreasing function. Hence, P_e decreases as the difference $x_{01}(T) - x_{02}(T)$ becomes greater and the rms voltage σ becomes smaller. The optimum filter has to maximize the ratio $\frac{X_{01}(T) - X_{02}(T)}{\sigma}$ in such a manner that the probability of error P_e is minimum.

Evaluation of Transfer function for the optimum filter :-

The transfer function of the optimum filter in such way that it will maximize the ratio $\frac{x_{01}(T) - X_{02}(T)}{\sigma}$. The difference signal $x_{01}(T) - X_{02}(T) = x_0(T)$.

This means that the optimum filter has to maximize the ratio $\frac{x_0(T)}{\sigma}$.

$$\left(\frac{S}{N}\right) = \frac{x_0^2(T)}{\sigma^2}$$

where $\left(\frac{S}{N}\right)_0$ is known as the signal to noise power ratio.

Thus, we are maximizing square of $[\frac{NO(T)}{\sigma})$.

In this equation, $x_0^2(T)$ is the normalized signal power in 1Ω load.

Further $\sigma^2 = n_0^2(T) = E[n_0^2(T)]$ is normalized noise power because mean value of noise is zero. Thus, the optimum filter has to maximize the ratio.

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{n_0^2(T)} \text{ (or) } \frac{x_0^2(T)}{\sigma^2} \text{ (or) } \frac{x_0^2(T)}{E[n_0^2(T)]}.$$

Again, let $X_0(f)$ be fourier transform of $x_0(t)$. If $x(f)$ is the fourier transform of input difference signal $x(t)$.

then $X_0(f) = H(f).X(f)$.

$H(f)$ is the transfer function of optimum filter.

$x_0(T)$ is by taking inverse fourier transform of $X_0(f)$

$$x_0(T) = \text{IFT}[x_0(f)] = \int_{-\infty}^{\infty} X_0(f).e^{j2\pi fT}.df.$$

then. $$X_0(T) = \int_{-\infty}^{\infty} H(f).X(f)\, e^{j2\pi fT}.df.$$

The normalized noise power can be obtained by integrating the power spectral density.

$$\sigma^2 = \int_{-\infty}^{\infty} S_{no}(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 S_{ni}(f).df.$$

Then $$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \frac{\left[\int_{-\infty}^{\infty} H(f).X(f)\, e^{j2\pi fT}\, df\right]^2}{\int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f).df.}$$

The schwarz's inequality which states that

$$\left[\int_{-\infty}^{\infty} \emptyset_1(x)\,\emptyset_2(x)\,dx\right]^2 \leq \int_{-\infty}^{\infty} |\emptyset_1(x)|^2.dx \int_{-\infty}^{\infty} |\emptyset_2(x)|^2.dx$$

$\theta_1(f) = \sqrt{S_{ni}(f)}\ H(f)$ $\theta_2(f) = \frac{1}{\sqrt{S_{ni}(f)}}\ X(f)\ e^{j2\pi fT}$

then $\left(\frac{S}{N}\right)_0 = \frac{\left|\int_{-\infty}^{\infty}\theta_1(f)\cdot\theta_2(f)\,df\right|^2}{\int_{-\infty}^{\infty}|\theta_1(f)|^2 df.}$

Applying schwarz's inequality of equation.

$$\left(\frac{S}{N}\right)_0 \leq \frac{\int_{-\infty}^{\infty}\cancel{|\theta_1(f)|^2 df}\cdot\int_{-\infty}^{\infty}|\theta_2(f)|^2 df}{\int_{-\infty}^{\infty}\cancel{|\theta_1(f)|^2 df}}$$

$$\left(\frac{S}{N}\right)_0 \leq \int_{-\infty}^{\infty}|\theta_2(f)|^2 df \leq \int_{-\infty}^{\infty}\frac{1}{S_{ni}(f)}\left[X(f)\,e^{j2\pi fT}\right]^2 df.$$

putting the value of $\theta_2(f)$ in the above equation

$|x(f)\ e^{j2\pi fT}|^2 = |X(f)|^2$ since $|e^{j2\pi fT}| = 1$

then $\gamma \leq \int_{-\infty}^{\infty}\frac{|x(f)|^2}{S_{ni}(f)}\cdot df.$

The signal to noise ratio is.

$$\gamma_{max} = \int_{-\infty}^{\infty}\frac{|x(f)|^2}{S_{ni}(f)}\cdot df.$$

This is possible when equality applies in schwarz's in equality. The equality is possible only if $\theta_1(f) = K.\theta_2^*(f)$ putting the values of $\theta_1(f)$ and $\theta_2(f)$.

$$\sqrt{S_{ni}(f)}\ H(f) = K.\frac{1}{\sqrt{S_{ni}(f)}}\ x^*(f)\ e^{-j2\pi fT}$$

$$H(f) = K.\frac{x^*(f)}{S_{ni}(f)}\ e^{-j2\pi fT}.$$

Finally the transfer function.

$$\boxed{H(f) = K.\frac{X^{*}(f)}{S_{ni}(f)} e^{-j2\pi fT}}$$

The Signal to noise ratio

$$\rho_{max} = \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^{2}_{max}$$

$$\boxed{\rho_{max} = \int_{-\infty}^{\infty} \frac{|x(f)|^{2}}{S_{ni}(f)} . df}$$

Matched filter :-

In optimum filter we considered generalized gaussian noise. when this noise is white gaussian noise, then the optimum filter is called matched filter.

→ For the white gaussian noise the power spectral density is given as.

$$S_{ni}(f) = \frac{N_0}{2}.$$

Calculation of impulse Response for the matched filter :-

The transfer function of the optimum filter is.

$$H(f) = K.\frac{X^{*}(f)}{S_{ni}(f)} e^{-j2\pi fT}$$

The transfer function of the matched filter

$$H(f) = K.\frac{X^{*}(f)}{\frac{N_0}{2}} e^{-j2\pi fT}$$

$$H(f) = \frac{2K}{N_0} X^{*}(f) e^{-j2\pi fT}$$

From the property of fourier transform,

$$X^*(f) = X(-f).$$

then

$$H(f) = \frac{2K}{N_0} X(-f) e^{-j2\pi fT}$$

The impulse response of a matched filter can be evaluated by taking inverse fourier transform.

$$h(t) = IFT[H(f)] = IFT\left[\frac{2K}{N_0} X(-f) e^{-j2\pi fT}\right]$$

The inverse fourier transform of $x(-f)$ is $x(-t)$.

$$FT[x(-t)] = x(-f) \text{ and}$$

$$FT[x(T-t)] = X(-f) e^{-j2\pi fT}$$

By this properties of fourier transform.

$$h(t) = \frac{2K}{N_0} x(T-t)$$

$$x(t) = x_1(t) - x_2(t)$$

$$\boxed{h(t) = \frac{2k}{N_0}[x_1(T-t) - x_2(T-t)]}$$

This gives the required impulse response of the matched filter.

probability of Error of the matched filter :-

The probability of error of matched filter

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right]$$

In the above equation

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \int_{-\infty}^{\infty} \frac{|x(f)|^2}{S_{ni}(f)} \cdot df$$

In the above equation $S_{ni}(f) = \frac{N_0}{2}$.

then

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \int_{-\infty}^{\infty} \frac{|x(f)|^2}{\frac{N_0}{2}} \cdot df$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} |x(f)|^2 \cdot df.$$

Signal power :-

By using the Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |x(f)|^2 \cdot df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_0^T x^2(t) \cdot dt$$

Since $x(t)$ exists from 0 to T only. we know that.

$$x(t) = x_1(t) - x_2(t).$$

$$\int_{-\infty}^{\infty} |x(f)|^2 \cdot df = \int_0^T [x_1(t) - x_2(t)]^2 \cdot dt.$$

$$= \int_0^T [x_1^2(t) + x_2^2(t) - 2x_1(t)\, x_2(t)]\, dt$$

$$= \int_0^T x_1^2(t)\, dt + \int_0^T x_2^2(t) dt - 2\int_0^T x_1(t)\, x_2(t)\, dt$$

here $\int_0^T x_1^2(t)\, dt = E_1$ & $\int_0^T x_2^2(t)\, dt = E_2$

and $\int_0^T x_1(t)\, x_2(t) = E_{12}$ represents energy due to auto-correlation between $x_1(t)$ and $x_2(t)$.

If we select $x_1(t) = -x_2(t)$, then the energies are equal.

$$E_1 = E_2 = -E_{12} = E.$$

then $\int_{-\infty}^{\infty} |x(f)|^2 df = [E + E - 2(-E)]$

$$= 4E.$$

Error probability :-

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \cdot 4E$$

$$= \frac{8E}{N_0}$$

$$= 2\sqrt{2} \cdot \sqrt{\frac{E}{N_0}}$$

putting this value of $\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]$ in equation then probability of error of matched filter as.

$$\boxed{P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0}}}$$

Error probability of ASK :-

In Amplitude shift keying (ASK). some number of carrier cycles are transmitted to send '1' and no signal is transmitted for binary '0'.

Binary '1' = $x_1(t) = \sqrt{2P} \cos(2\pi f_c t)$ and

Binary '0' = $x_2(t) = 0$.

$$P_e = \frac{1}{2} \operatorname{erfc} \left[\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}\right]$$

In this

$$\left[\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}\right]^2 = \int_{-\infty}^{\infty} \frac{(x(f))^2}{S_{ni}(f)} \cdot df \Rightarrow \frac{2}{N_0} \int_{-\infty}^{\infty} (x(f))^2 df$$

$$S_{ni}(f) = \frac{N_0}{2}$$

By using the power theorem.

$$\int_{-\infty}^{\infty} |x(f)|^2 . df = \int_{-\infty}^{\infty} x^2(t). dt$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{2}{N_0} \int_{-\infty}^{\infty} x^2(t)\, dt$$

in this $x(t) = x_1(t) - x_2(t)$.

$x_1(t)$ for binary '1'

$x_1(t) = \sqrt{2P} \cos(2\pi f_c t)$

$x_2(t) = 0$ for binary '0' & $x(t)$ is present from 0 to T.

$$\frac{2}{N_0} \int_0^T x_1^2(t)\, dt = \frac{2}{N_0} \int_0^T \left[\sqrt{2P} \cos(2\pi f_c t)\right]^2 dt$$

$$= \frac{4P}{N_0} \int_0^T \cos^2(2\pi f_c t)\, dt = \frac{4P}{N_0} \int_0^T \frac{1 + \cos(4\pi f_c t)}{2}\, dt$$

$$= \frac{\cancel{4}^{2} P}{N_0} \times \frac{1}{\cancel{2}} \int_0^T 1 + \cos(4\pi f_c t)\, dt = \frac{2P}{N_0} \int_0^T 1\, dt + \frac{2P}{N_0} \int_0^T \cos(4\pi f_c t)\, dt$$

$$= \frac{2P}{N_0} \left\{ [t]_0^T + \left[\frac{\sin 4\pi f_c t}{4\pi f_c}\right]_0^T \right\}$$

$$= \frac{2P}{N_0} \left\{ T + \underbrace{\frac{\sin 4\pi f_0 T}{4\pi f_0}}_{0} \right\}$$

(∵ $f_0 T$ = integer no. of cycles]
(∴ $f_0 T = k$)
$\sin 4\pi k$ (k is interger

$$= \frac{2PT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{2PT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right] = \sqrt{\frac{2PT}{N_0}}$$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}} \cdot \frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]$$

$$= \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{2PT}{N_0}}\right)$$

$$= \frac{1}{2}\,\text{erfc}\left[\frac{1}{2} \cdot \sqrt{\frac{E}{N_0}}\right]$$

$$= \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E}{4N_0}}\right.$$

Probability of error for coherently detected BPSK

In BPSK. Binary '1' $\Rightarrow x_1(t) = \sqrt{2P}\cos(2\pi f_c t)$

Binary '0' $\Rightarrow x_2(t) = -\sqrt{2P}\cos(2\pi f_c t)$.

The probability of error

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right]$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{2}{N_0}\int_0^T x^2(t)\,dt$$

$$x(t) = x_1(t) - x_2(t)$$

$$\therefore\ x_2(t) = -x_1(t).$$

then $x(t) = x_1(t) - (-x_1(t))$

$$= 2x_1(t)$$

then

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{2}{N_0}\int_0^T (2x_1(t))^2\,dt$$

$$= \frac{2}{N_0}\int_0^T 4.\,x_1^2(t)\,dt$$

$$= \frac{8}{N_0}\int_0^T x_1^2(t)\,dt$$

$$\int_0^T x_1^2(t)\,dt = \frac{8}{N_0}\int_0^T \left(\sqrt{2P}\cos(2\pi f_c t)\right)^2 dt = \frac{8}{N_0}\int_0^T 2P\cos^2(2\pi f_c t)\,dt$$

$$= \frac{8}{N_0}\int_0^T 2P\left[\frac{1+\cos(4\pi f_c t)}{2}\right]dt = \frac{16P}{N_0}\times\frac{1}{2}\int_0^T 1+\cos(4\pi f_c t)\,dt$$

$$= \frac{8P}{N_0}\int_0^T 1\,dt + \frac{8P}{N_0}\int_0^T \cos(4\pi f_c t)\,dt$$

$$= \frac{8P}{N_0}\left[T\right]_0^T + \frac{8P}{N_0}\left[\frac{\sin 4\pi f_c t}{4\pi f_c}\right]_0^T$$

$$= \frac{8PT}{N_0} + \underbrace{\frac{8P}{N_0}\left[\frac{\sin 4\pi f_c T}{4\pi f_c}\right]}_{\to \text{'0'}}$$

$$= \frac{8PT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{8PT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right] = \sqrt{\frac{8PT}{N_0}} = \sqrt{\frac{8E}{N_0}}$$

$$\sqrt{\frac{8E}{N_0}} = 2\sqrt{2}\cdot\sqrt{\frac{E}{N_0}}$$

Then : $P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{8E}{N_0}}\right]$

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E}{N_0}}}$$

probability of error for coherently detected BFSK :-

In FSK Binary '1' ⇒ $x_1(t) = \sqrt{2P}\cos(2\pi f_c + \Omega)t$

Binary '0' ⇒ $x_2(t) = \sqrt{2P}\cos(2\pi f_c - \Omega)t$

$$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right]$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right]^2 = \frac{2}{N_0}\int_{-\infty}^{\infty}(x(T))^2\,dt$$

$$x(T) = x_1(T) - x_2(T)$$

$$[x_1(T) - x_2(T)] = \{\sqrt{2P}\cos(2\pi f_c + \Omega)t - \sqrt{2P}\cos(2\pi f_c - \Omega)t\}$$

$$= \sqrt{2P}\left[\cos(2\pi f_c + \Omega)t - \cancel{2}P\cos(2\pi f_c - \Omega)t\right]$$

$$[x_1(t) - x_2(t)]^2 = 2P\left[\cos(2\pi f_c t + \Omega)t - \cos(2\pi f_c - \Omega)t\right]^2$$

$$\boxed{2\sin(x)\sin(y) = \cos(x - y) - \cos(x + y)}$$

$$[x_1(t) - x_2(t)]^2 = 2P\left[-2\sin 2\pi f_c t \sin\Omega t\right]^2$$

$$= 2P\left[4\sin^2 w_0 t \sin^2\Omega t\right]$$

$$= 2P\left[(2\sin^2 w_0 t)(2\sin^2\Omega t)\right]$$

$$2\sin^2(x) = 1 - \cos(2x)$$

$$[x_1(t) - x_2(t)]^2 = 2P\{(1 - \cos 2\Omega t)\cdot(1 - \cos 2w_0 t)\}$$

$$= 2P\{1 - \cos 2\Omega t - \cos 2w_0 t + \cos 2w_0 t \cos 2\Omega t\}$$

$$\boxed{\cos x \cdot \cos y = \frac{1}{2}\cos[x - y) + \cos(x + y]}$$

$$[x_1(t) - x_2(t)]^2 = 2P\left\{1 - \cos 2\Omega t - \cos 2\omega_0 t + \frac{1}{2}\left[\cos 2(\omega_0 - \Omega)t + \cos 2(\omega_0 + \Omega)t\right]\right\}$$

$$\int_0^T [x_1(t) - x_2(t)]^2 dt = \int_0^T 2P\left[1 - \cos 2\Omega t - \cos 2\omega_0 t + \frac{1}{2}\left[\cos 2(\omega_0 - \Omega)t + \cos 2(\omega_0 + \Omega)t\right]\right\} dt$$

$$= 2P\left[\int_0^T dt - \int_0^T \cos 2\Omega t\, dt - \int_0^T \cos 2\omega_0 t\, dt + \frac{1}{2}\int_0^T \cos 2(\omega_0 - \Omega)t\, dt + \frac{1}{2}\int_0^T \cos 2(\omega_0 + \Omega)t\, dt\right.$$

$$= 2P\left\{T - \frac{\sin 2\Omega T}{2\Omega} - \frac{\sin 2\omega_0 T}{2\omega_0} + \frac{1}{2}\frac{\sin 2(\omega_0 - \Omega)T}{2(\omega_0 - \Omega)} + \frac{1}{2}\frac{\sin 2(\omega_0 + \Omega)T}{2(\omega_0 + \Omega)}\right.$$

$$= 2PT\left\{1 - \frac{\sin 2\Omega T}{2\Omega} - \frac{\sin 2\omega_0 T}{2\omega_0 T} + \frac{1}{2}\frac{\sin 2(\omega_0 - \Omega)T}{2(\omega_0 - \Omega)T} + \frac{1}{2}\frac{\sin 2(\omega_0 + \Omega)T}{2(\omega_0 - \Omega)T}\right\}$$

The frequency shift 'Ω' is very small in comparision with the carrier frequency ω_0. Then the last three terms in the above equation will be of the form $\frac{\sin 2\omega_0 T}{2\omega_0 T}$

→ ω_0 is the angular frequency of the carrier signal and T is the period of one bit.

→ Actually many cycles of carrier are completed in one bit period. i.e $\omega_0 T >> 1$. Hence the ratio $\frac{\sin 2\omega_0 T}{2\omega_0 T}$ approaches to '0'. Therefore we can neglect last three terms.

$$\int_0^T [x_1(t) - x_2(t)]^2 dt = 2PT\left\{1 - \frac{\sin 2\Omega T}{2\Omega T}\right\}$$

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}}\right\}$$

$$\left[\frac{x_{01}[T] - x_{02}(T)}{2\sigma}\right]^2 = \frac{2}{N_0}\int_0^T [x_1(t) - x_2(t)]^2 dt$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{2}{N_0} \cdot 2PT \left\{1 - \frac{\sin 2\pi \Omega T}{2\Omega T}\right\}$$

$$= \frac{4PT}{N_0}\left\{1 - \frac{\sin 2\Omega T}{2\Omega T}\right\}$$

$$2\Omega T = \frac{3\pi}{2}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2 = \frac{4P.T}{N_0}\left[1 - \frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}}\right]$$

$$= \frac{4.84\,PT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right] = \sqrt{\frac{4.84PT}{N_0}}$$

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{2.\sqrt{2}} \cdot \sqrt{\frac{4.84PT}{N_0}}\right\}$$

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{0.6\,P.T}{N_0}}} \Rightarrow \boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{0.6PT}{N_0}}}$$

probability of error for QPSK :

The signal space Representation of QPSK observe that transmitted reference carriers are $\phi_1(t)$ and $\phi_2(t)$. All the signal vectors A, B, C & D are at 45° to these reference carriers. consider the receiver for QPSK signal. There are two correlators for two reference carriers. These two correlators are actually BPSK receivers error probability of BPSK, due to imperfect phase is given by

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b \cos^2\theta}{N_0}}$$

Hence error probability of correlator 1 is

$$P_{e1} = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b\cos^2\theta}{N_0}}$$

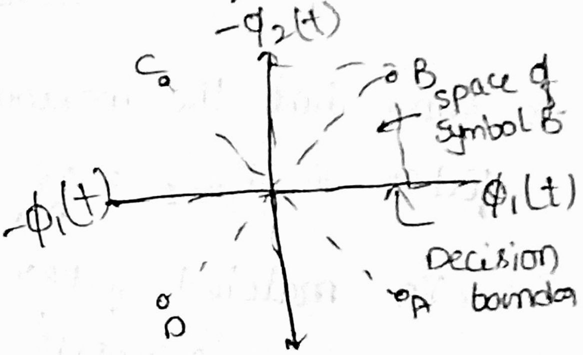
The error probability of correlator 2 is.

$$P_{e2} = P_{e1} = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b\cos^2\theta}{N_0}}$$

The correlators detect wrong symbol if phase shift of the carrier is more than 45°. Hence putting $\theta = 45^\circ$.

$$P_{e1} = P_{e2} = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b\cos^2(45)}{N_0}}$$

$$= \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{N_0}\times\frac{1}{2}}$$



Hence probability of getting correct symbol can be expressed as

$$P_e = (1 - P_{e1})(1 - P_{e2})$$

we know that $P_{e1} = P_{e2}$

$$P_c = 1 - 2P_{e1} + P_{e1}^2 \quad (\because P_{e1} \ll 1)$$

$$= 1 - 2P_{e1}$$

Then probability of error is given in terms of 'P_c' as.

$$P_e = 1 - P_c$$

$$= 1 - (1 - 2P_{e1}) = 2P_{e1}$$

Putting for P_{e1} from the equation.

$$P_e = 2 \times \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{2N_0}}.$$

Thus error probability of QPSK, $\boxed{P_e = \text{erfc}\sqrt{\frac{E_b}{2N_0}}}$

In this. $T_S = 2T_b$

$E_S = \frac{1}{2} A^2 . 2T_b$

then $E_S = \frac{1}{2} A^2 T_S$

$= \cancel{2}\left[\frac{1}{\cancel{2}} A^2 T_b\right]$

$= 2A^2 T_b$

$E_b = \frac{E_S}{2}.$

$E_S = 2E_b$

then $\boxed{P_e = \text{erfc}\sqrt{\frac{E_S}{4N_0}}}$

⇒ Show that the maximum signal to noise ratio of the matched filter is $P_{max} = \frac{2E}{N_0}$

In matched filter

$$\frac{S}{N} = \int_{-\infty}^{\infty} \frac{(x(f))^2}{S_{ni}(f)} df \qquad S_{ni}(f) = \frac{N_0}{2}$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} |x(f)|^2 df$$

by using power theorem.

$$\int_{-\infty}^{\infty} (x(f))^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = E.$$

then $P_{max} = \frac{2}{N_0} E$

$= \frac{2E}{N_0}$

$$\boxed{P_{max} = \frac{2E}{N_0}}$$

5 ④

unit - IV

INFORMATION THEORY

→ The information theory is related to the concepts of statistical properties of messages/sources, channels, noise interference etc. The information theory is used for mathematical modeling and analysis of the communication systems.

Information source :-

This the point where original information generates is called the information source can be classified as.

→ **Analog information source :-**

In Analog information source, information source is Analog in nature.

→ **Discrete information source :-**

Discrete information source consists of a sequence of discrete symbols or letters. Discrete symbols or information sources are characterized by the following parameters.

i) source alphapet
ii) symbol rate. (symbols/sec)
iii) source Alphabet probabilities.
iv) probabilistic dependence of symbols in a sequences.

mathematical Representation of source :-

Consider the source which emits the discrete symbols randomly from the set of fixed alphabet that is

$$X = [x_0, x_1, x_2 \,....\, x_{K-1}]$$

The various symbols in 'X' have probabilities of $P_0, P_1, P_2 \,....\, P_{K-1}$ etc.

which can be written as $P[X = x_K] = P_K$

$K = 0, 1, 2 \ldots K-1$

The set of probabilities is given as

$$\sum_{K=0}^{K-1} P_K = 1$$

If the probability of X_K is P_K.

If $P_K = 0$ then symbol is impossible.

If $P_K = 1$ then symbol is possible

Definition of information :-

Let us consider a communication system which transmits messages $m_1, m_2, m_3, \ldots$ with probabilities of occurence $P_1, P_2, P_3 \ldots$ The amount of information transmitted through the message m_K with probability P_K is given as

Amount of Information $I_K = \log_2 \left(\frac{1}{P_K}\right)$

In the above equation given as.

$$\log_2 \left(\frac{1}{P_K}\right) = \frac{\log_{10} (1/P_K)}{\log_{10} 2}.$$

Properties of Information ;-

1. If there is more uncertainty about message, information carried is also more.
2. If receiver knows the message being transmitted, the amount of information carried is zero.
3. If I_1 is the information carried by message m_1 and I_2 is the information carried by message m_2, then

amount of information carried combinedly due to m_1 & m_2 is $I_1 + I_2$.

4. If there are $M = 2^N$ equally likely messages then amount of information carried by each message will be N bits.

1. A source produces one of four possible symbols during each interval having probabilities $P(x_1) = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$, $P(x_3) = P(x_4) = \frac{1}{8}$. obtain the Information content of each of these symbols.

Ans :-
$$I_K = \log_2\left(\frac{1}{P_K}\right)$$

1. $P(x_1) = \frac{1}{2}$ then $I(x_1) = \log_2\left(\frac{1}{1/2}\right) = \log_2(2)$

$= 1$ bit & 1 binit

2. $P(x_2) = \frac{1}{4}$ then $I(x_2) = \log_2\left(1/1/4\right) = \log_2(4)$

$= 2$ bits

3. $P(x_3) = P(x_4) = \frac{1}{8}$ then $I(x_i) = \log_2\left(\frac{1}{1/8}\right)$

$= \log_2(8)$

$= 3$ bits.

Entropy (Average information)

Entropy is defined in terms of bits per symbol. Bit or binit is the abbreviation for binary digit.

Consider there m- different messages. let these messages be $m_1, m_2, m_3 \ldots m_m$ and they have probabilities of occurrence as $P_1, P_2, P_3 \ldots P_m$. suppose that a sequence of L messages is transmitted. Then if L is very very large.

$P_1 L$ messages of m_1 are transmitted,
$P_2 L$ messages of m_2 are transmitted,
⋮
$P_m L$ messages of m_m are transmitted.

Hence the information due to message m_1 will be

$$I_1 = \log_2\left(\frac{1}{P_1}\right)$$

Since there are $P_1 L$ number of messages of m_1, the total. information due to all message of m_1 will be.

$$I_1 \text{ (total)} = P_1 L \log_2\left(\frac{1}{P_1}\right)$$

similarly $I_2 \text{ (total)} = P_2 L \log_2\left(\frac{1}{P_2}\right)$.

Thus the total information carried due to the sequence of L messages will be;

$$I\text{(total)} = I_1\text{ (total)} + I_2\text{(total)} + \cdots + I_m\text{(Total)}.$$

$$\therefore I\text{ (total)} = P_1 L \log_2\left(\frac{1}{P_1}\right) + P_2 L \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m L \log_2\left(\frac{1}{P_m}\right)$$

The average information per message will be.

$$\text{Average information}(H) = \frac{\text{Total information}}{\text{Number of messages}} = \frac{I\text{(total)}}{L}.$$

then Entropy $(H) = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m \log_2\left(\frac{1}{P_m}\right)$

we can write above equation using Σ sign.

then Entropy $: H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)$

$$\boxed{H = \sum_{K=1}^{m} P_K \log_2\left(\frac{1}{P_K}\right)}$$

P_1L messages of m_1 are transmitted,

P_2L messages of m_2 are transmitted,

⋮

P_mL messages of m_m are transmitted.

Hence the information due to message m_1 will be

$$I_1 = \log_2\left(\frac{1}{P_1}\right)$$

since there are P_1L number of messages of m_1, the total information due to all message of m_1 will be.

$$I_1\,(\text{total}) = P_1L\,\log_2\left(\frac{1}{P_1}\right)$$

similarly $I_2\,(\text{total}) = P_2L\,\log_2\left(\frac{1}{P_2}\right)$.

Thus the total information carried due to the sequence of L messages will be;

$$I\,(\text{total}) = I_1\,(\text{total}) + I_2\,(\text{total}) + \dots + I_m\,(\text{Total}).$$

$$\therefore\ I\,(\text{total}) = P_1L\,\log_2\left(\frac{1}{P_1}\right) + P_2L\,\log_2\left(\frac{1}{P_2}\right) + \dots + P_mL\,\log_2\left(\frac{1}{P_m}\right)$$

The average information per message will be.

$$\text{Average information}(H) = \frac{\text{Total information}}{\text{Number of messages}} = \frac{I(\text{total})}{L}.$$

then $\text{Entropy}\,(H) = P_1\log_2\left(\frac{1}{P_1}\right) + P_2\log_2\left(\frac{1}{P_2}\right) + \dots + P_m\log_2\left(\frac{1}{P_m}\right)$

we can write above equation using Σ sign.

then $\text{Entropy} = H = \sum_{K=1}^{M} P_K\,\log_2\left(\frac{1}{P_K}\right)$

$$\boxed{H = \sum_{K=1}^{m} P_K\,\log_2\left(\frac{1}{P_K}\right)}$$

P_1L messages of m_1 are transmitted.

P_2L messages of m_2 are transmitted.

⋮

P_mL messages of m_m are transmitted

Hence the information due to message m_1 will be

$$I_1 = \log_2\left(\frac{1}{P_1}\right)$$

Since there are P_1L number of messages of m_1, the total information due to all message of m_1 will be.

$$I_1(\text{total}) = P_1L \log_2\left(\frac{1}{P_1}\right)$$

Similarly $I_2(\text{total}) = P_2L \log_2\left(\frac{1}{P_2}\right)$.

Thus the total information carried due to the Sequence of L messages will be.

$$I(\text{total}) = I_1(\text{total}) + I_2(\text{total}) + \cdots + I_m(\text{Total})$$

$$\therefore I(\text{total}) = P_1L \log_2\left(\frac{1}{P_1}\right) + P_2L \log_2\left(\frac{1}{P_2}\right) + \cdots + P_mL \log_2\left(\frac{1}{P_m}\right)$$

The average information per message will be

$$\text{Average information}(H) = \frac{\text{Total information}}{\text{number of messages}} = \frac{I(\text{total})}{L}$$

then $\text{Entropy}(H) = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m \log_2\left(\frac{1}{P_m}\right)$

we can write above equation using Σ sign

then $\text{Entropy}: H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)$

$$\boxed{H = \sum_{K=1}^{m} P_K \log_2\left(\frac{1}{P_K}\right)}$$

Properties of Entropy :-

1. entropy is zero if the event is sure or it is impossible.

   $H = 0$ if $P_K = 0$ or 1.

2. when $P_K = \frac{1}{M}$ for all the 'M' symbols, then the symbols are equally likely. for such source entropy is given as.

   $H = \log_2 M$.

3. upper bound on entropy is given as. $H_{max} = \log_2 M$.

1. calculate entropy when $P_K = 0$ and when $P_K = 1$

$$H = \sum_{K=1}^{M} P_K \log_2 \left(\frac{1}{P_K}\right).$$

Since $P_K = 1$.

$$H = \sum_{K=1}^{M} 1 . \log_2 \left(\frac{1}{1}\right) = \sum_{K=1}^{M} \frac{\log_{10}(1)}{\log_{10}(2)} = 0.$$

Since $P_K = 0$

$$H = \sum_{K=1}^{M} 0 . \log_2 \left(\frac{1}{0}\right) = 0.$$

Thus entropy is zero ~~both~~ for both $P_K = 0$ or 1.

2. Show that if there are 'M' number of equally likely messages, then Entropy of the source is $\log_2 M$.

we know that for 'M' number of equally likely messages, probability is $P = \frac{1}{M}$.

This probability is same for all 'M' messages.

$$P_1 = P_2 = P_3 = P_4 = \cdots P_m = \frac{1}{M}.$$

Entropy is given as.

$$H = \sum_{K=1}^{m} P_K \log_2\left(\frac{1}{P_K}\right)$$

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m \log_2\left(\frac{1}{P_M}\right)$$

putting the probabilities is $\frac{1}{M}$.

$$H = \frac{1}{M}\log_2(M) + \frac{1}{M}\log_2(m) + \cdots + m \log_2(M)$$

In the above equation there are 'M' number of terms in summation. Hence after adding these terms above equation becomes

$$H = \cancel{M} \times \frac{1}{\cancel{M}} \log_2(M)$$

$$\boxed{H = \log_2 M}$$

3. Prove that the upper bound on entropy is given as. $H_{max} \leq \log_2 M$. Here M is the number of messages emitted by the source.

To prove the above property, we will use the following property of natural logarithm.

$$\ln x \leq x - 1 \quad \text{for } x \geq 0$$

let us consider any two probability distributions $\{P_1, P_2, \ldots P_M\}$ and $\{q_1, q_2, \ldots q_m\}$ on the alphabet $X = \{x_1, x_2 \cdots x_m\}$ of the discrete memoryless source. Then let us consider the term $\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right)$. This term can be written as.

$$\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right) = \sum_{K=1}^{m} P_K \frac{\log_{10}\left(\frac{q_K}{P_K}\right)}{\log_{10} 2}.$$

multiply the R.H.S by $\log_{10} e$ and rearrange terms.

$$\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right) = \sum_{K=1}^{m} P_K \frac{\log_{10} e}{\log_{10} e} \cdot \frac{\log_{10}\left(\frac{q_K}{P_K}\right)}{\log_{10} 2}$$

$$= \sum_{K=1}^{m} P_K \frac{\log_{10} e}{\log_{10} 2} \cdot \frac{\log_{10}\left[\frac{q_K}{P_K}\right)}{\log_{10} e}$$

$$= \sum_{K=1}^{m} P_K \log_2 e \cdot \log_e\left(\frac{q_K}{P_K}\right)$$

$$\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right) = \log_2 e \sum_{K=1}^{m} P_K \ln\left[\frac{q_K}{P_K}\right) \quad \left(\because \log_e\left(\frac{q_K}{P_K}\right) = \ln\left(\frac{q_K}{P_K}\right)\right)$$

$$\ln\left(\frac{q_K}{P_K}\right) \leq \left[\frac{q_K}{P_K} - 1\right]$$

$$\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right) \leq \log_2 e \sum_{K=1}^{m} P_K \left(\frac{q_K}{P_K} - 1\right]$$

$$\leq \log_2 e \sum_{K=1}^{m} (q_K - P_K)$$

$$\leq \log_2 e \left[\sum_{K=1}^{m} q_K - \sum_{K=1}^{M} P_K\right] \quad \left[\sum_{K=1}^{m} q_K = 1,\right. \quad \sum_{K=1}^{m} P_K = 1$$

$$\sum_{K=1}^{m} P_K \log_2\left(\frac{q_K}{P_K}\right) \leq 0.$$

Now, let us consider that $q_K = \frac{1}{m}$ for all K.

$$\sum_{K=1}^{m} P_K \left[\log_2 q_K + \log_2 \frac{1}{P_K}\right] \leq 0$$

$$\sum_{K=1}^{m} P_K \log_2 q_K + \sum_{K=1}^{M} P_K \log_2 \frac{1}{P_K} \leq 0$$

$$\sum_{K=1}^{m} P_K \log_2 \frac{1}{P_K} \leq -\sum_{K=1}^{m} P_K \log_2 q_K.$$

$$\sum_{K=1}^{m} P_K \log_2 \frac{1}{P_K} \leq \sum_{K=1}^{M} P_K \log_2 \frac{1}{q_K}. \quad (\because q_K = \frac{1}{M})$$

$$\sum_{K=1}^{m} P_K \log_2 \frac{1}{P_K} \leq \sum_{K=1}^{M} P_K \log_2 (M)$$

$$\leq \log_2 M . \sum_{K=1}^{M} P_K$$

since $\sum_{K=1}^{m} P_K = 1$.

$$\sum_{K=1}^{m} P_K \log_2 \left(\frac{1}{P_K}\right) \leq \log_2 M$$

The L.H.S of above equation is entropy.

$$H \leq \log_2 M.$$

This is the proof of upper bound on entropy. And maximum value of entropy is

$$\boxed{H_{max} = \log_2 M}$$

1. A source generates four messages m_0, m_1, m_2 and m_3 with probabilities $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}$ and $\frac{1}{4}$ respectively. The successive messages emitted by the source are statistically independent. calculate entropy of the source.

$$H = \sum_{K=1}^{m} P_K \log_2 \left(\frac{1}{P_K}\right)$$

$$H = P_1 \log_2 \left(\frac{1}{P_1}\right) + P_2 \log_2 \left(\frac{1}{P_2}\right) + P_3 \log_2 \left(\frac{1}{P_3}\right) + P_4 \log_2 \left(\frac{1}{P_4}\right)$$

$$= \frac{1}{3} \log_2 (3) + \frac{1}{6} \log_2 (6) + \frac{1}{4} \log_2 (4) + \frac{1}{4} \log_2 (4)$$

= 1959 bits/message.

Information Rate :-

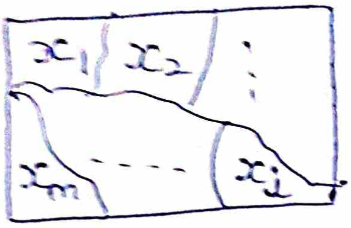
The information rate is represented by R. and it is given as

Information rate = Symbol rate x source entropy.
(Bits|sec) = (symbols or mess/sec) x (Bits|message).

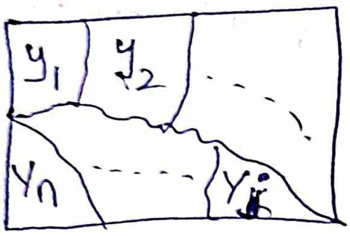
$$R = rH.$$

marginal and conditional Entropies :-

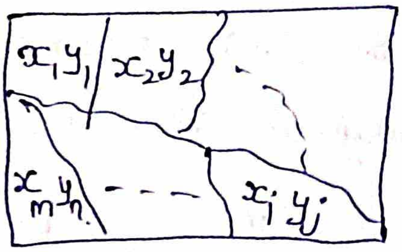
To study the behaviour of the communication system, the concept of two-dimensional probability scheme is necessary.



① Sample space S_1



② Sample space S_2



③ Sample space of joint probabilities.

The figure shows 3 the sample space that consists of joint occurrence of two events x_i and y_j. In such situation we have four sets of probability schemes.

i) probability of X, $P(x_i)$
ii) Probability of Y, $P(y_j)$
iii) joint probability of XY $P(x_i, y_j)$
iv) conditional probability $P(x_i/y_j)$ and $P(y_j/x_i)$

If we add all joint probabilities for fixed x_i we get

$$P(y_j) = \sum_{i=1}^{m} P(x_i, y_j)$$

If we add all joint probabilities for fixed ~~x_i~~ y_j

$$P(x_i) = \sum_{j=1}^{n} P(x_i, y_j).$$

Joint Entropy:-

The joint entropy represents entropy of joint occurance of two or more events.

→ The joint Entropy is given as.

$$H(x,Y) = H(Y,x) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

→ Here $H(x,Y)$ reperesents entropy of joint occurance of X and Y.

marginal Entropy:-

when the entropy of individual event is evaluated from joint probabilities of the events, it is called marginal entropy.

$$H(x) = \sum_{i=1}^{M} P(x_i)$$

putting for $P(x_i)$ from the above equation.

$$H(x) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} P(x_i, y_j)\right] = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j]$$

Similarly.

$$H(Y) = \sum_{j=1}^{n} P(Y_j) \qquad P(Y_j) = \sum_{i=1}^{m} P(x_i, y_j)$$

$$= \sum_{j=1}^{n}\left[\sum_{i=1}^{m} P(x_i, y_j)\right] = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)$$

Conditional entropy:-

The conditional Entropy $H(x/Y)$ represents uncertainty of X, on average when Y is known. Similarly the conditional entropy $H(Y/x)$ represents uncertainty of Y, on averages, when X is transmitted.

Conditional entropy indicates the information lost across the noisy channel. The mathematical equation of $H(x/y)$ and $H(y/x)$ are

$$H(x/y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i/y_j) \log_2 \frac{1}{P(x_i|y_j)}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 P(x_i|y_j)$$

and $$H(y/x) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(y_j|x_i)}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 P(y_j|x_i).$$

* prove that.

$H(x,y) = H(x/y) + H(y).$

$H(x,y) = H(y/x) + H(x)$

<u>solution</u> i) $H(x,y) = H(x/y) + H(y)$.

$$H(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 P(x_i, y_j)$$

From Probability theory.

$$P(AB) = P\left(\frac{A}{B}\right). P(B)$$

$P(x_i, y_j) = P\left(\frac{x_i}{y_j}\right). P(y_j)$ then

$$H(x,y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \underbrace{\log_2 \left[P\left(\frac{x_i}{y_j}\right). P(y_j)\right]}_{\log_2\left[P(x_i/y_j) \cdot P(y_j)\right] = \left[\log_2 P\left(\frac{x_i}{y_j}\right) + \log_2 P(y_j)\right]}$$

$$H(x,Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\,\underbrace{\log_2 P(x_i,y_j)}\left[\log_2 P\left(\frac{x_i}{y_j}\right) + \log_2 P\left(y_j\right)\right]$$

$$H(x,Y) = -\sum_{j=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\,\log_2 P(x_i/y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i y_j)\,\log_2 P(y_j)$$

$$H(x,Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\,\log_2 \frac{1}{P(x_i/y_j)} - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i y_j]\,\log_2 P(y_j)]$$

$$= H(x/Y) - \sum_{j=1}^{m}\left\{\sum_{i=1}^{n} P(x_i,y_j)\right\}\log_2 P(y_j)$$

$$= H(x/y) - \sum_{j=1}^{m} P(y_j)\,\log_2 P(y_j)$$

$$= H(x/Y) + \sum_{j=1}^{m} P(y_j)\,\log_2 \frac{1}{P(y_j)}$$

$$= H(x/Y) + H(Y).$$

$$\boxed{H(x,Y) = H(x/Y) + H(Y)}$$

ii) $H(x,Y) = H(Y/x) + H(x).$

$$H(x_i,y_j) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\,\log_2 P(x_i,y_j)$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\,\log_2\left[P\left(\frac{y_j}{x_i}\right)\cdot P(x_i)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i y_j)\,\log_2 P\left(\frac{y_j}{x_i}\right) - \sum_{i=1}^{m}\left[\sum_{j=1}^{n} P(x_i,y_j)\right]\log_2 P(x_i)$$

$$= H\left(\frac{Y}{x}\right) - \sum_{i=1}^{m} P(x_i)\,\log_2 P(x_i).$$

$$= H\left(\frac{Y}{x}\right) + \sum_{i=1}^{m} P(x_i)\,\log_2 \frac{1}{P(x_i)} = H\left(\frac{Y}{x}\right) + H(x)$$

mutual information:-

The mutual information is defined as the amount of information transferred when x_i is transmitted and y_j is received. It is represented by $I(x_i, y_j)$ and given as,

$$I(x_i, y_j) = \log_2 \frac{P(x_i/y_j)}{P(x_i)} \text{ bits.}$$

Here $I(x_i, y_j)$ is the mutual information.

$P(x_i/y_j)$ is the conditional probability that x_i was Transmitted and y_j is received.

$P(x_i)$ is the probability of symbol x_i for transmission.

The average mutual information is represented by $I(X; Y)$. It is calculated in bits/symbol. The average mutual information is defined as the amount of source information gained per received symbol.

$$I(X; Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \, I(x_i, y_j).$$

Thus $I(x_i, y_j)$ is weighted by joint probabilities $P(x_i, y_j)$ over all the possible joint events.

$$I(X; Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

* Properties of mutual information:-

1) The mutual information of the channel is symmetric.

$$I(X; Y) = I(Y; X).$$

$$I(Y; X) = I(X; Y).$$

2. The mutual information can be expressed in terms of entropies of channel input & output and conditional entropies.

$I(x;Y) = H(x) - H(x/Y)$ $\quad$ $H(x|y)$ & $H(Y|x)$ are conditional entropies.

$= H(Y) - H(Y/x)$.

3. The mutual information is always positive. $I(x;Y) \geq 0$.

4. The mutual information is related to the joint Entropy $H(x,Y)$ by following relation.

$I(x;Y). = H(x) + H(Y) - H(x,Y)$.

→ prove that the mutual information of the channel is symmetric.

1) $I(X;Y) = I(Y;x)$

Solution :- let us consider some standard relationships from probability theory.

$P(x_i, y_j) = P\left(\frac{x_i}{y_j}\right) \cdot P(y_j)$ and

$P(x_i, y_j) = P\left(\frac{y_j}{x_i}\right) P(x_i)$

$P\left(\frac{x_i}{y_j}\right) \cdot P(y_j) = P\left(\frac{y_j}{x_i}\right) P(x_i)$

$\therefore \frac{P(x_i|y_j)}{P(x_i)} = \frac{P(y_j|x_i)}{P(y_j)}$

The average mutual information is given as.

$$I(x;Y) = \sum_{i=1}^{m} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i|y_j)}{P(x_i)}.$$

Hence, we can write as.

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\log_2 \frac{P(y_j|x_i)}{P(y_j)}.$$

$$\boxed{I(X;Y) = I(Y;X)}$$

→2) prove the following relationships.

i) $I(X;Y) = H(X) - H(X/Y)$

ii) $I(X;Y) = H(Y) - H(Y/X)$.

Solution:- $H(X/Y)$ is the conditional entropy.

$$H(X/Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i,y_j)\log_2 \frac{1}{P(x_i|y_j)}$$

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log_2 \frac{P(x_i|y_j)}{P(x_i)}$$

let us write the above equation as.

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log_2 \left[\frac{1}{P(x_i)}\cdot P(x_i|y_j)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log_2 \frac{1}{P(x_i)} - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log_2 \frac{1}{P(x_i|y_j)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log_2 \frac{1}{P(x_i)} - H(X/Y).$$

$$\boxed{\sum_{j=1}^{n} P(x_i\,y_j) = P(x_i)}$$

$$= \sum_{i=1}^{m} P(x_i)\log_2 \frac{1}{P(x_i)} - H(X/Y) \Rightarrow H(X) - H(X/Y).$$

$$\boxed{I(x:Y) = H(x) - H(x/Y)}$$

ii) $\boxed{I(x:Y) = H(Y) - H(Y/x)}$

$$H\left(\frac{Y}{x}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(y_j|x_i)}$$

$$I(Y;x) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{P(y_j|x_i)}{P(y_j)}$$

The above equation can be written as.

$$I(Y:x) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \left(\frac{1}{P(y_j)} \cdot P(y_j|x_i)\right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \left[\log_2 \frac{1}{P(y_j)} + \log_2 P[y_j|x_i)\right]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(y_j)} + \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 P(y_j/x_i)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(y_j)} - \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(y_j/x_i)}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i y_j) \log_2 \frac{1}{P(y_j)} - H(Y/x)$$

$$= \sum_{j=1}^{n} P(y_j) \log_2 \frac{1}{P(y_j)} - H(Y/x) \qquad \sum_{i=1}^{m} P(x_i, y_j) = P(y_j)$$

$$= H(Y) - H(Y/x).$$

* prove that the mutual information is always positive.

3) $I(x;Y) \geq 0.$

Solution :- mutual information is given by.

$$I(x;Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i | y_j)}{P(x_i)}$$

$$P(x_i / y_j) = \frac{P(x_i \, y_j)}{P(y_j)} \qquad P(x_i, y_j) = P\left(\frac{x_i}{y_j}\right) . P(y_j)$$

putting the value of $P(x_i | y_j)$ in above equation.

$$I(x;Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i ; y_j)}{P(y_j) P(x_i)}$$

$$I(x,Y) = - \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(y_j) P(x_i)}{P(x_i, y_j)}$$

This equation can be written as

$$- I(x,Y) = + \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log \frac{P(y_j) P(x_i)}{P(x_i, y_j)}$$

$$\sum_{K=1}^{m} P_K \log_2 \left(\frac{q_K}{P_K}\right) \leq 0$$

If we can P_K be $P(x_i \, y_j)$ and q_K be $P(x_i) P(y_j)$. Both P_K and q_K are two probability distributions on same alphabet.

$$-I(x;Y) \leq 0$$

$$I(x;Y) \geq 0.$$

It says that mutual information is always non-negative.

+ prove the $I(X;Y) = H(X) + H(Y) - H(X,Y)$

$$H(X,Y) = H(X/Y) + H(Y)$$

$$H(X/Y) = H(X,Y) - H(Y).$$

from the 2nd properties

$$I(X;Y) = H(X) - H(X/Y)$$

$$I(X;Y) = H(X) - \left(H(X,Y) - H(Y)\right)$$

$$= H(X) + H(Y) - H(X,Y)$$

Discrete memoryless channels :-

→ The discrete communication channel has input x and output Y. Both x and Y are the random variables. The channel is discrete when both x and Y are discrete.

→ The channel is called memory less (zero memory) when current output depends only on current input.

→ The Transition probabilities of the channel can be represented by a matrix as follows

$$P(Y/X) = \begin{bmatrix} P(Y_1|x_1) & P(Y_2|x_1) & \dots & P(Y_m|x_1) \\ P(Y_1|x_2) & P(Y_2|x_2) & \dots & P(Y_m|x_2) \\ \vdots & \vdots & & \vdots \\ P(y_1|x_n) & P(y_2|x_n) & \dots & P(Y_m|x_n) \end{bmatrix}$$

The above matrix has the size of n×m. it is called the channel matrix. It is called the channel matrix, noise matrix or probability transition matrix.

$$P(Y_1|x_1) + P(y_2|x_1) + \dots + P(Y_m|x_1) = 1$$

This is applicable to other rows also.

$$\sum_{j=1}^{m} P(y_j|x_i) = 1$$

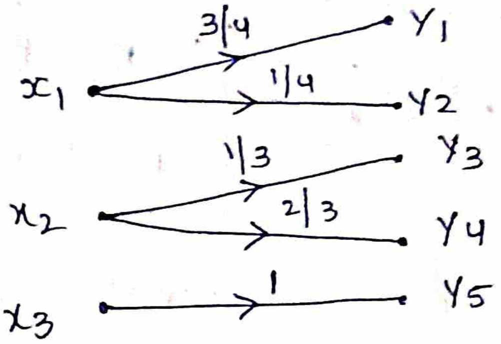
Types of channels :-

There are some special type of channels with their own channel matrices.

1. Loss less channel :-

A channel described by a channel matrix with only one non-zero element in each column is called a loss less channel. An example of a loss less channel is given as.

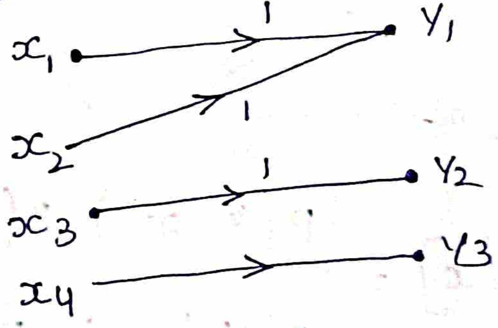
$$P[Y/x] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



2. Deterministic channel :-

A channel described by a channel matrix with only one non-zero element in each row is called a Deterministic channel. An example of a Deterministic channel is given as.

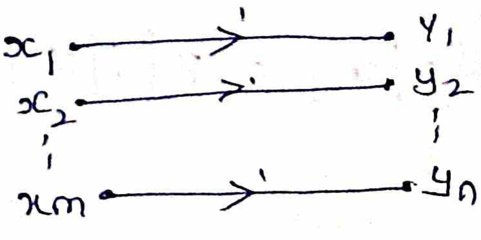
$$P[Y/x] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3. Noiseless channel :-

A channel is called noise less if it is both loss less and deterministic. A Noise less channel matrix has only, one element in each row and in each column and this element is unity. The input and output alaphapets are of the same size m = n for the noise less channel.

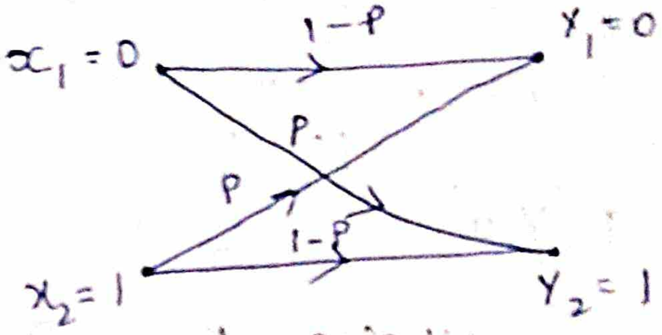
$$P[Y/x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Binary symmetric Channel (BSC)

The binary symmetric channel (BSC) is defined by the below channel diagram. A BSC channel has two inputs ($x_1=0, x_2=1$) and two outputs ($Y_1=0, Y_2=1$) This channel is symmetric because the probability of receiving a 1 if a 0 is sent is the same as the probability of receiving a 0 if a 1 is sent.

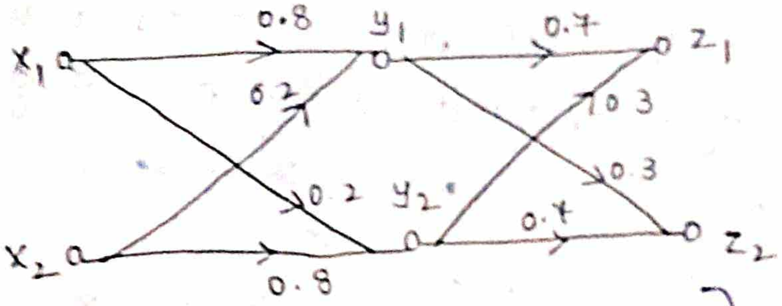
$$P[Y|x] = \begin{bmatrix} 1-P & P \\ P & 1-P \end{bmatrix}$$



1. Two BSC's are connected in cascade as shown in fig.

i) find the channel matrix of resultant channel.

ii) find $P(Z_1)$ and $P(Z_2)$ if $P(x_1)=0.6$ and $P(x_2)=0.4$.



Solution :-

$$P[Y/x] = \begin{bmatrix} P(Y_1/x_1) & P(Y_2|x_1) \\ P(Y_1/x_2) & P(Y_2|x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P[Z/Y] = \begin{bmatrix} P(Z_1|Y_1) & P(Z_2/Y_1) \\ P(Z_1|Y_2) & P(Z_2/Y_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Hence the resultant channel matrix is given as

$$P[Z/x] = P(Y/x) . P(Z/Y)$$

$$P[z/x] = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \\ x_3y_1 + x_4y_3 & x_3y_2 + x_4y_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

ii) To obtain $P(z_1)$ and $P(z_2)$

$$P(x;z) = P\left(\frac{z}{x}\right) \cdot P(x)$$

$$P(z) = P\left(\frac{z}{x}\right) \cdot P(x)$$

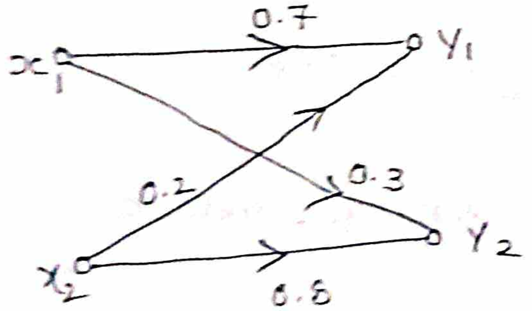
$$= \begin{bmatrix} P[x_1] & P[x_2] \end{bmatrix} \cdot \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}_{2\times2}$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}_{2\times1}$$

$$= \begin{bmatrix} 0.524 & 0.476 \end{bmatrix}$$

Then $P(z_1) : 0.524$, $P(z_2) = 0.476$.

2. Find the channel capacity in below figure.



Solution:- Three steps to solve this problem.

1. Form the channel matrix
2. Form Pai matrix and obtain values of ai.
3. obtain capacity by muroga's method.

step 1 :-

$$P = \begin{bmatrix} P[Y_1|x_1] & P(y_2|x_1) \\ P(Y_1|x_2) & P[y_2|x_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$$

step 2 :- PQ matrix and values of Q.

$$\begin{bmatrix} P(Y_1|x_1) & P(y_2|x_1) \\ P(y_1|x_2) & P(y_2|x_2) \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P(y_1|x_1) \log_2 P(y_1|x_1) + P\left(\frac{Y_2}{x_1}\right) \log_2 P\left(\frac{y_2}{x_1}\right) \\ P(y_1|x_2) \log_2 P(y_1|x_2) + P\left(\frac{y_2}{x_2}\right) \log_2 P\left(\frac{y_2}{x_2}\right) \end{bmatrix}$$

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0.7 \log_2 0.7 + 0.3 \log_2 0.3 \\ 0.2 \log_2 0.2 + 0.8 \log_2 0.8 \end{bmatrix}$$

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -0.88 \\ -0.722 \end{bmatrix}$$

$$0.7 Q_1 + 0.3 Q_2 = -0.88$$

$$0.2 Q_1 + 0.8 Q_2 = -0.722$$

solving the above equation.

$Q_1 = -0.9748$ and $Q_2 = -0.6588$.

step3 :- channel capacity by muroga's method.

$$C = \log_2 \left[2^{Q_1} + 2^{Q_2} \right]$$

$$= \log_2 \left[2^{-0.9748} + 2^{-0.6588} \right]$$

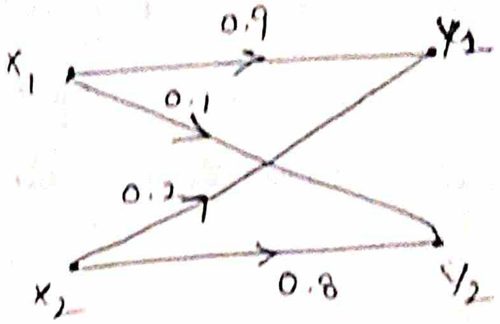
$$= 0.1918 \text{ bits/symbol.}$$

3. Given a binary channel shown in figure.

i) find the channel matrix of the channel.

ii) find $P(Y_1)$ & $P(Y_2)$ when $P(X_1) = P(X_2) = 0.5$

iii) find the joint probabilities $P(x_1, y_2)$ & $P(x_2, y_1)$ when $P(X_1) = P(X_2) = 0.5$.



Solution :-

i). channel matrix :-

$$P[Y/x] = \begin{bmatrix} P(Y_1/x_1) & P(Y_2/x_2) \\ P(Y_1/x_2) & P(Y_2/x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

ii) we know that $P(xY) = P\left(\frac{Y}{x}\right) P(x)$

substituting all the values, we get.

$$P(xY) = P\left(\frac{Y}{x}\right) \cdot P(x)$$

$$P(Y) = P\left(\frac{Y}{x}\right) \cdot P(x)$$

$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \left[P(x_1) . P(X_2)\right]$$

$$= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \end{bmatrix}$$

iii). $P[x, Y] = \left[P[x]\right]_d \cdot \left(P[Y/x]\right)$

$$= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.45 & 0.05 \\ 0.1 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} Y_1|X_1 & Y_2|x_1 \\ Y_1|x_2 & Y_2|x_2 \end{bmatrix}$$

$$= \begin{bmatrix} P(x,Y_1) & P(x_1,y_2) \\ P(x_2,Y_1) & P(x_2,Y_2) \end{bmatrix} \Rightarrow \begin{array}{l} P(x_1,y_2) = 0.05 \\ P(x_2,Y_1) = 0.1 \end{array}$$

4. A channel has the following channel matrix.

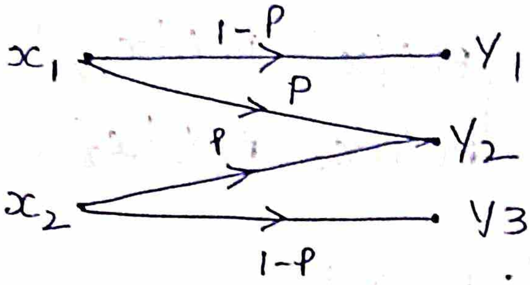
$$P\left[\frac{Y}{X}\right] = \begin{bmatrix} 1-P & P & 0 \\ 0 & P & 1-P \end{bmatrix}$$

i) Draw the channel diagram

ii) If the source has equally likely outputs, compute the probabilities associated with the channel outputs for $P = 0.2$.

Solution :-

i. Channel diagram.



ii) $P[Y] = P[X] \cdot P\left[\frac{Y}{X}\right]$

$$= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix}$$

5. For a channel matrix gives below, find $I(X;Y)$ given the input symbols occur with equal probability.

$$P[Y/X] = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Solutions :- The input symbols occur with equal probabilities

$$P[X] = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

probabilities of output symbols.

$$P[Y] = P\left[\frac{Y}{X}\right] \cdot P(X)$$

$$= \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \overset{a_{11}}{\frac{1}{3}} & \overset{a_{12}}{\frac{1}{3}} & \overset{a_{13}}{\frac{1}{3}} \end{bmatrix}$$

ii) To obtain entropy of output $H(Y)$.

$$P(Y_1) = P(y_2) = P(y_3) = \frac{1}{3}$$

$$P_1 = P_2 = P_3 = \frac{1}{3}$$

$$H(Y) = \sum_{i=1}^{s} P(Y_i) \log_2 \frac{1}{P(y_i)}$$

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right)$$

$$= 3 \times \frac{1}{3} \log_2 (3)$$

$= 1.585$ bits | symbol.

iii) To obtain $P(x_i, y_j)$

$$P(x_i, y_j) = P\left(\frac{y_j}{x_i}\right) \cdot P(x_i)$$

From the given matrix

→ $P(Y_1/x_1) = P(Y_2/x_2) = P(Y_3/x_3) = 0.6$ then.

$$P(X_1 y_1) = P(X_2 y_2) = P(X_3 Y_3) = 0.6 \times \frac{1}{3} = 0.2$$

→ $P(Y_2|x_1) = P(Y_3/X_1) = P(Y_1/x_2) = P(Y_3|x_2) = P(Y_1|x_3) = P(Y_2/x_3) = 0.2$

then $P(X_1 y_2) = P(x_1 y_3) = P(x_2 y_1) = P(x_2 y_3) = P(X_3 Y_1) = P(x_3 Y_2) = \frac{0.2}{3} = 0.0667.$

$$P[x_i, y_j] = \begin{bmatrix} P(Y_1|x_1) \times P(x_1) & P(Y_2/x_1) \times P(x_1) & P(Y_3/x_1) \times P(x_1) \\ P(Y_1|x_2) \times P(x_2) & P(Y_2/x_2) \times P(x_2) & P(Y_3/x_2) \times P(x_2) \\ P(Y_1/x_3) \times P(x_3) & P(Y_2/x_3) \times P(x_3) & P(Y_3|x_3) \times P(x_3) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.0667 & 0.0667 \\ 0.0667 & 0.2 & 0.0667 \\ 0.0667 & 0.0667 & 0.2 \end{bmatrix}$$

iv) conditional entropy.

$$H(Y/x) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{1}{P(Y_j/x_i)}$$

$$= 3 \times \left[0.2 \log_2 \left(\frac{1}{0.6}\right)\right] + 6 \times \left(0.0667 \log_2 \left(\frac{1}{0.2}\right)\right]$$

$= 1.37$ bits / symbol.

mutual information.

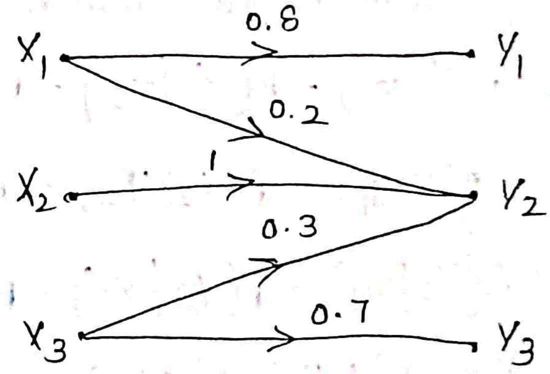
$$I(Y;x) = H(Y) - H\left(\frac{Y}{x}\right)$$

$$= 1.585 - 1.37$$

$$= 0.215 \text{ bits / symbol.}$$

5. A discrete source transmits messages x_1, x_2 & x_3 with the probabilities 0.3, 0.4 and 0.3. The source is connected to the channel given in fig. calculate the all the entropies.

solutions:-



$$P[Y|x] = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 0.7 \end{bmatrix}$$

$$P[x] = [0.3 \quad 0.4 \quad 0.3]$$

$P[x_1] = 0.3$, $P[x_2] = 0.4$, $P[x_3] = 0.3$.

we know the values of $P[x]$ then

$$H(x) = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)} = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.3 \log_2\left(\frac{1}{0.3}\right)$$

$$= 0.52 + 0.52 + 0.52 = 1.568 \text{ bits / symbol.}$$

from the figure.

$$P(x,y) = P\left[Y/x\right] \cdot P(x) = \begin{bmatrix} 0.8 \times 0.3 & 0.2 \times 0.3 & 0 \\ 0 & 1 \times 0.4 & 0 \\ 0 & 0.3 \times 0.3 & 0.3 \times 0.7 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} Y_1 & X_2 & Y_3 \end{array} \\ \begin{bmatrix} 0.24 & 0.06 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0.09 & 0.21 \end{bmatrix} \end{array}$$

$P(y_1) = 0.24$

$P(y_2) = 0.06 + 0.4 + 0.09 = 0.55$

$P(y_3) = 0.21$

then $H(Y) = \sum_{j=1}^{m} P(Y_j) \log_2\left(\frac{1}{P(Y_j)}\right) = \sum_{j=1}^{3} P(y_j) \log_2\left(\frac{1}{P(y_j)}\right)$

$= 0.24 \log_2\left(\frac{1}{0.24}\right) + 0.55 \log_2\left(\frac{1}{0.55}\right) + 0.21 \log_2\left(\frac{1}{0.21}\right)$

$= 1.441$ bits / message

$H(x,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2\left(\frac{1}{P(x_i y_j)}\right)$

$= - \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 (P(x_i y_j))$

$= - \sum_{i=1}^{3} \sum_{j=1}^{3} P(x_i y_j) \log_2 P(x_i y_j)$

$= - [0.24 \log_2 (0.24) + 0.06 \log_2 (0.06) + 0.4 \log_2 (0.4) + 0.09 \log_2 (0.09) +$
$0.21 \log_2 (0.21)$

$= 2.053$ bits / message.

$H(x/Y) = H(x,Y) - H(Y)$. (or) $\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(x_i | y_j)}$

$= 2.053 - 1.441$

$= 0.612$ bits / message.

$H(Y/x) = H(x,Y) - H(x)$ (or) $\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(y_j | x_i)}$.

$= 2.053 - 1.571$

$= 0.482$ bits / message

6. Consider a telegraph source having two symbols dot and dash. The dot duration is 0.2 sec and the dash duration is 3 times of the dot duration. The probability of the dot's occuring is twice that of dash, and time between symbols is 0.2 seconds. Calculate the information rate of the telegraph source. Consider the string of 1200 symbols.

The probability of dash = P.

probability of dot = 2P.

$$P + 2P = 1 \Rightarrow 3P = 1 \Rightarrow P = \frac{1}{3}.$$

Then $P_{(dash)} = \frac{1}{3}$ and $P_{(dot)} = 2 \times \frac{1}{3} = \frac{2}{3}$.

Entropy $H = \sum_{K=1}^{m} P_K \log_2 \left(\frac{1}{P_K} \right) = \sum_{K=1}^{2} P_K \log_2 \left(\frac{1}{P_K} \right)$.

$$= \frac{1}{3} \log_2 (3) + \frac{2}{3} \log_2 \left[\frac{3}{2} \right) = 0.9183 \text{ bits / symbol.}$$

To calculate average symbol rate :-

Dot duration T_{dot} = 0.2 sec.

dash duration T_{dash} = 0.6 sec (3 × 0.2 sec).

Duration between symbols : 0.2 sec.

Number of dots = $1200 \times \frac{2}{3} = 800$

Number of dash : $1200 \times \frac{1}{3} = 400$.

Total time = Dots duration + dash duration + (1200 × Time between symbols).

$$= (800 \times 0.2) + (400 \times 0.6) + (1200 \times 0.2)$$

$$= 640 \text{ sec.}$$

Hence average symbol rate $r = \frac{1200}{T} = \frac{1200}{640} = 1.875$ symbols / sec.

The information rate

$R = rH$

$= 0.9183 \times 1.875$

$= 1.7218$ bits / sec.

7. Consider a discrete memory less source with source alphabet $\{s_0, s_1, s_2)$ and source statistics $\{0.7, 0.15, 0.15\}$.

i) calculate the entropy of the source.

ii) calculate the entropy of second order extension of the source.

i) $H(x) = \sum_{i=1}^{3} P_i \log_2 \frac{1}{P_i}$

$= 0.7 \log_2\left(\frac{1}{0.7}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right)$

$= 0.36 + 0.41 + 0.41 = 1.1805$ bits/symbols

ii) Now, the entropy of second order extension of the source can be evaluated as under.

$$H(x^n) = n \cdot H(x).$$

For second order then $n=2$

$$H(x^2) = 2 \cdot H(x) = 2 \times 1.1805 = 2.361 \text{ bits/symbol}$$

8. The probabilities of the five possible outcomes of an experiment are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}$. Determine the entropy and information rate if there are 16 outcomes per second. $(r = 16)$

9. A discrete source emits one of five symbols once every millisecond with probabilities $1/2, 1/4, 1/8, 1/16$ and $1/16$ respectively. Determine the source Entropy and information rate.

Solution :- $R = H \times r$ $\quad H = \sum_{k=1}^{5} P(x_i) \log_2 \frac{1}{P(x_i)} = 1.875$ bits/symbols

$$r = \frac{1}{T_b} = \frac{1}{10^{-3}} = 1000 \text{ symbols/sec.}$$

$$R = 1000 \times 1.875 = 1875 \text{ bits/sec.}$$

10. A Transmitter has an alphabet of 4 letters $[x_1, x_2, x_3, x_4]$ & the receiver has an alphabet of 3 letters $[y_1, y_2, y_3]$. The joint probability matrix is

$$P(x,y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.05 & 0 \\ x_2 & 0 & 0.25 & 0 \\ x_3 & 0 & 0.15 & 0.05 \\ x_4 & 0 & 0.05 & 0.15 \end{array}$$

Solution :

$$P(X,Y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ x_1 & 0.3 & 0.05 & 0 \\ x_2 & 0 & 0.25 & 0 \\ x_3 & 0 & 0.15 & 0.05 \\ x_4 & 0 & 0.05 & 0.15 \end{array}$$

$P(X) = [0.35 \quad 0.25 \quad 0.2 \quad 0.2]$

$P(Y) = [0.3 \quad 0.5 \quad 0.2]$

$$H(X) = -\sum_{i=1}^{4} P(x_i)\log_2 P(x_i)$$

$= -[0.35 \log(0.35) + 0.25 \log(0.25) + [0.2 \log_2(0.2) \times 2]]$

$= 1.96$ bits / message

$$H(Y) = -\sum_{j=1}^{3} P(y_j)\log_2 P(y_j)$$

$= -[0.3 \log_2(0.3) + 0.5 \log_2(0.5) + 0.2 \log_2(0.2)]$

$= 1.49$ bits / message

$$H(X,Y) = -\sum_{i=1}^{4}\sum_{j=1}^{3} P(x_i, y_j)\log_2(P(x_i, y_j))$$

$= -[0.3 \log_2(0.3) + [3 \times 0.05 \log_2(0.05)] + 0.25 \times \log_2(0.25) +$

$[2 \times 0.05 \log_2(0.05)]]$

$= 2.49$ bits / message

$H(X/Y) = H(X,Y) - H(Y) = 2.49 - 1.49 = 1.00$ bits / message

$H(Y/X) = H(X,Y) - H(X) = 2.49 - 1.96 = 0.53$ bits / message.

Then mutual information

$I(X:Y) = H(X) + H(Y) - H(XY)$

$= 1.96 + 1.49 - 2.49$

$= 3.45 - 2.49 = 0.96$ bits / symbol

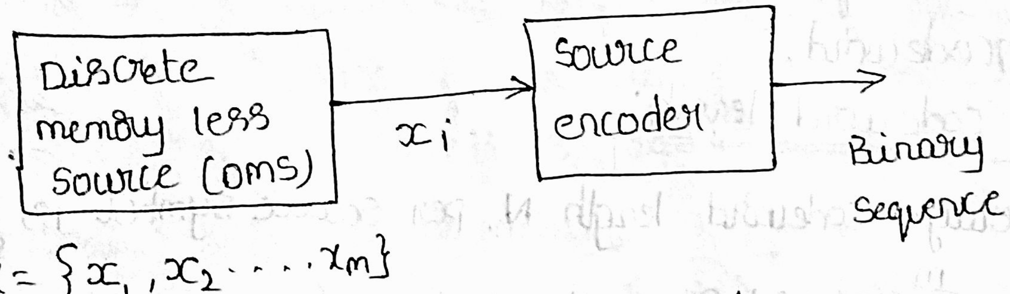
unit - V
Source Coding

Source coding theorem :- (shanon's first Theorem).

→ If the messages have different probabilities and they are assigned same number of binary digits, then information carrying capability of binary PCM is not completely utilized.

→ But if all the messages have same probabilities, then maximum information rate is possible with binary PCM coding. The device which performs source coding is called source encoder.

→ A conversion of the output of a discrete memoryless source (DMS) into a sequence of binary symbols is called source coding.



Block diagram for source coding.

→ Efficient source encoders can be designed with use the statical properties of the source. For example, the message occuring frequently can be assigned short codewords, where as messages which occurs rarely are assigned long codewords. Such coding is called variable length coding.

→ The efficient source encoder should satisfy following requirements.

1. The codewords generated by the encoder should be binary in nature
2. The source code should be unique in nature.

The source coding theorem states that for a discrete memoryless source of entropy H, the average codeword length N for any distortionless source encoding is bounded as.

$$\boxed{N \geq H}$$

1. Code word length :-

The source encoder assigns codewords to the symbols. let X be a DMS with finite entropy H(x) and an alphabet $\{x_1, \ldots . x_m)$. with corresponding probabilities of ocurrence $P(x_i)$ $(i = 1, 2, 3 \ldots m)$. let the binary codeword assigned to symbol x_i by the encoder having length n_i, measured in bits.

→ The length of a codeword is the number of binary digits in the codeword.

2. Average code word length:-

The average codeword length N, per source symbol is given by

$$N = \sum_{i=1}^{m} P(x_i) n_i$$

The parameter 'N' represents the average number of bits per source symbol used in the source coding process.

3. coding efficiency;-

The efficiency of the source code is given by

$$\eta = \frac{H}{N} = \frac{\text{Entropy}}{\text{Average number of bits.}}$$

$$\boxed{H = \sum_{k=1}^{M} P_k \log_2 \left(\frac{1}{P_k}\right)} \Rightarrow H = -\sum_{k=1}^{m} P_k \log_2 (P_k).$$

4. Code Redundancy :-

It is the measure of redundancy of bits in the encoded message sequence.

Redundancy (r) = 1 - code efficiency

$= 1 - \eta$.

Redundancy should be as low as possible.

5. Code variance :-

variance of the code is given as $\sigma^2 = \sum_{k=0}^{L-1} P_k (n_k - \bar{N})^2$

$N \rightarrow$ average codeword length.

$n_k \rightarrow$ The number of bits assigned to k^{th} symbol.

$P_k \rightarrow$ probability of k^{th} symbol.

L-1 or M → the number of symbols.

variance is the measure of variability in codeword lengths. variance should be as small as possible.

→ Entropy coding :- (Data compaction)

Data compaction : It is also called lossless data compression. It is basically a source encoding technique that has two aspects.

1. It is efficient interms of average number of bits per symbol,
ii) Original data can be reconstructed without loss of any information.

Examples of Entropy coding.

1. Shannon - Fano coding.
2. Huffman - coding.
   i) huffman - binary coding.
   ii) huffman - Ternary coding.
   iii) huffman - Quarternary coding.

Shannon - Fano coding :-

Algorithm :-

- → list the source symbols in order of decreasing probability.
- → partition the set into two sets that are as close to each as possible, and assign 0 to the upper set, '1' to the lower set.
- → continue this process, each time partitioning the sets with as nearly equal probabilities as possible until further partitioning is not possible.

problems :-

1. A DMS X has six symbols. x_1 to x_6 with $P(x_1) = 0.30$, $P(x_2) = 0.25$, $P(x_3) = 0.20$, $P(x_4) = 0.12$, $P(x_5) = 0.08$, and $P(x_6) = 0.05$. construct a shanaon - Fano code for x. and find out the code efficiency and code Redundancy.

Solution :-

message x_i	probability. $P(x_i)$	step 1	step 2	step 3	step. 4	code	length n_i
x_1	0.30	0	0			00	2
x_2	0.25	0	1			01	2
x_3	0.20	1	0			10	2
x_4	0.12	1	1	0		110	3
x_5	0.08	1	1	1	0	1110	4
x_6	0.05	1	1	1	1	1111	4

$$\text{code efficiency} = \eta = \frac{H}{L}$$

$$H = \sum_{K=1}^{m} P_K \log_2\left(\frac{1}{P_K}\right) \Rightarrow -\sum_{K=1}^{6} P_K \log_2(P_K)$$

$$= -(0.30 \log_2(0.30) + 0.25 \log_2(0.25) + 0.20 \log_2(0.20) + 0.12 \log_2(0.12) + 0.08 \log_2(0.08) + 0.05 \log_2(0.05)).$$

$H = 2.36$ bit | symbol.

Average codeword length

$$N \text{ or } L = \sum_{i=1}^{m} P_i n_i \Rightarrow \sum_{i=1}^{6} P_i n_i$$

$$= (0.30)(2) + 0.25(2) + 0.20(2) + 0.12(3) + 0.08(4) + 0.05(4).$$

$$= 0.6 + 0.5 + 0.4 + 0.36 + 0.32 + 0.2$$

$= 2.38$ bits | symbol.

then $\eta = \frac{2.36}{2.38} = 0.99. \Rightarrow 99\%.$

$r = 1 - \eta = 0.01 = 1.$

2. Apply the shannon - Fano coding produce for the following message. and find out the effiency and redundancy.

$$[X] = [x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8]$$

$$[P] = [1/4 \; 1/8 \; 1/16 \; 1/16 \; 1/16 \; 1/4 \; 1/16 \; 1/8]$$

solution:-

message X	probability $P(x_i)$	Step 1	step 2	step 3		Encoded message	length (n_i)
x_1	0.25	0	0			00	2
x_6	0.25	0	1			01	2
x_2	0.125	1	0	0		100	3
x_8	0.125	1	0	1		101	3
x_3	0.0625	1	1	0	0	1100	4
x_4	0.0625	1	1	0	1	1101	4
x_5	0.0625	1	1	1	0	1110	4
x_7	0.0625	1	1	1	1	1111	4.

Entropy $H(X) = \sum_{K=1}^{m} P_K \log_2 \left(\frac{1}{P_K}\right)$

$= -\sum_{K=1}^{8} P_K \log_2 (P_K)$

$= -\Big(0.25 \log_2(0.25) + 0.25 \log_2(0.25) + 0.125 \log_2 (0.125) + 0.125 \log_2 (0.125). + \big(0.0625 \log_2 (0.0625) \times 4\big)\Big).$

$=$ 2.75 bits | Symbol.

Average codeword length.

$\bar{N} = \sum_{i=1}^{8} P_i n_i$

$= (0.25 \times 2) + (0.25 \times 2) + 2(0.125 \times 3) + 4(0.0625 \times 4)$

$=$ 2.75 bits | symbol.

efficiency $\eta = \frac{2.75}{275} = 1 \Rightarrow 100\%.$

$R = 1 - \eta = \underline{0}$

3. Apply the shannon - Fano coding for the following message.

$[X] : [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6, \ x_7]$

$[P] : [0.4, 0.2, 0.12 \ 0.08 \ 0.08 \ 0.08 \ 0.04].$

4. Apply shannon - Fano coding for the following message.

$[X] : [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8]$

$[P] = [16/32 \ 4/32 \ 4/32 \ 2/32 \ 2/32 \ 2/32 \ 1/32 \ 1/32]$

5. A DMS X have five equally symbols.

i) construct a Shanon - Fano Code for X, and calculate the effiency of the Code

ii) construct another shannon - Fano code and Compare the results.

Huffman coding :- (Huffman binary coding).

Algorithm :-

→ list the source symbols in order of decreasing probability.

→ combine the probabilities of the two symbols having the lowest probabilities and reorder the resultant probabilities. This step is called reduction. The same procedure is repeated until there are two ordered probabilities remaining.

→ Start encoding with the last reduction, which consist of exactly two ordered probabilities. Assign 0 as the first digit in the codewords for all the source symbols associated with the first probability ; assign 1 to the second probability.

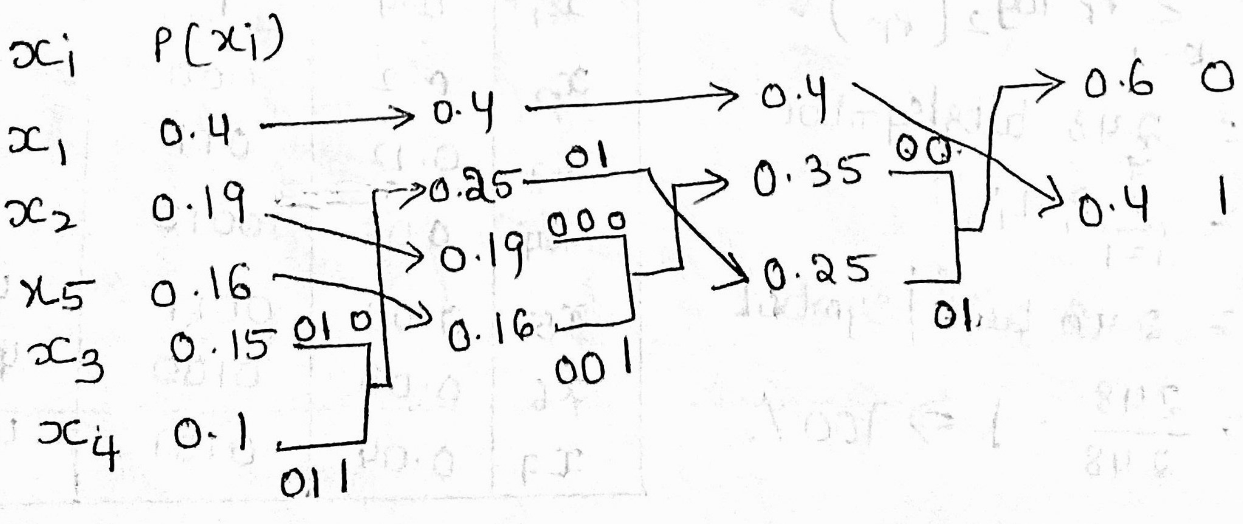
→ Now go back and assign '0' and '1' to the second digit for the two probabilities that were combined in the previous reduction step, retaining all assignments made in step 3.

→ Keep regressing this way until the first column is reached.

problems :-

1. A DMS X has five symbols x_1 to x_5 with $P(x_1) = 0.4$, $P(x_2) = 0.19$, $P(x_3) = 0.15$, $P(x_4) = 0.1$ $P(x_5) = 0.16$. Construct a Huffman code and find out the efficiency and Redundancy.

Solution :-



x_i	$P(x_i)$	Code	length n_i
x_1	0.4	1	1
x_2	0.19	000	3
x_5	0.16	001	3
x_3	0.15	010	3
x_4	0.1	011	3

$$H = -\sum_{k=1}^{5} P_K \log_2(P_K)$$

$= 2.15$ bit / symbol.

$$N = \sum_{i=1}^{5} P_i n_i$$

$= 2.2$ bits / symbol

$\eta = 0.977 \Rightarrow 97.7\%$.

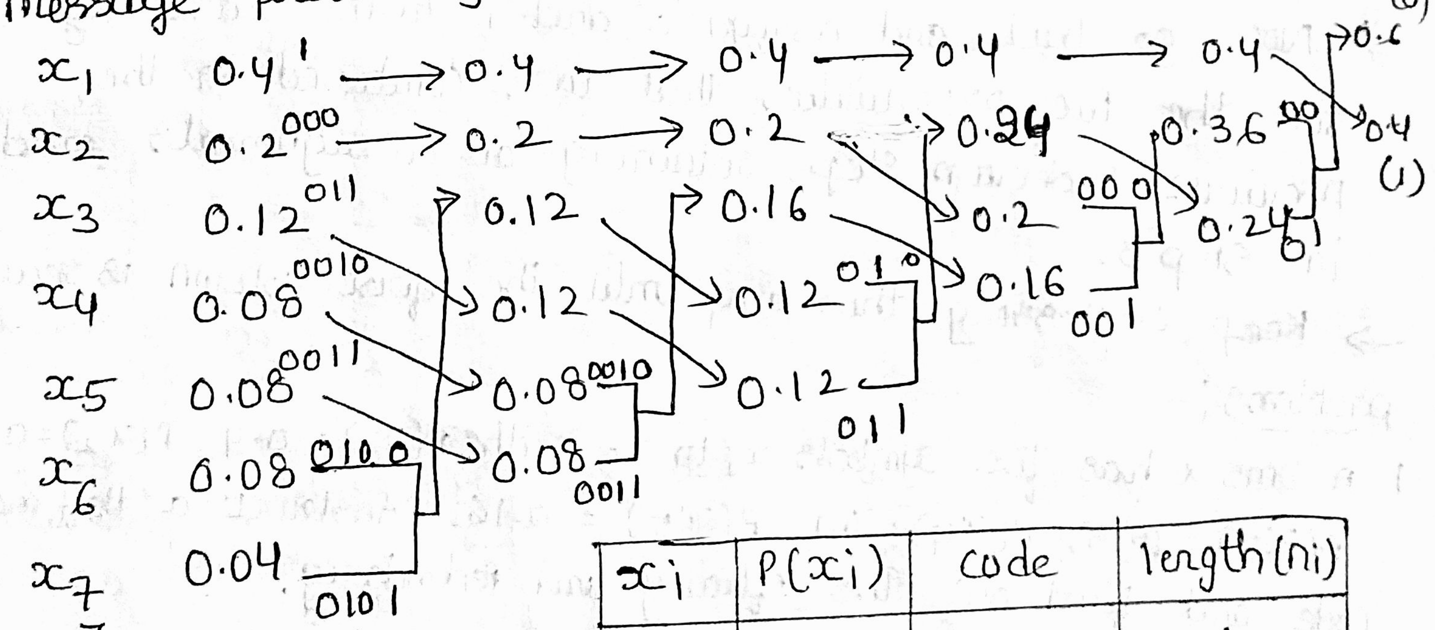
2. Apply the huffman coding produre for the following messages.

$[x] = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7]$

$[P] = [0.4 \ 0.2 \ 0.12 \ 0.08 \ 0.08 \ 0.08 \ 0.04]$

solution:-

message probability.



$$H = \sum_{k=1}^{7} P_K \log_2\left(\frac{1}{P_K}\right)$$

$= 2.48$ bits/symbol

$$N = \sum_{i=1}^{7} P_i n_i$$

$= 2.48$ bits / symbol

$$\eta = \frac{2.48}{2.48} = 1 \Rightarrow 100\%.$$

x_i	$P(x_i)$	code	length (n_i)
x_1	0.4	1	1
x_2	0.2	000	3
x_3	0.12	011	3
x_4	0.08	0010	4
x_5	0.08	0011	4
x_6	0.08	0100	4
x_7	0.04	0101	4.

3. A. DMS have five symbols. S_0, S_1, S_2, S_3 and S_4, characterized by probability distribution as 0.4 0.2, 0.1, 0.2 and 0.1 respectively. show that by using huffman coding the average codeword length is same but the variance varies in the following cases. i) as high as possible.
ii) As low as possible.

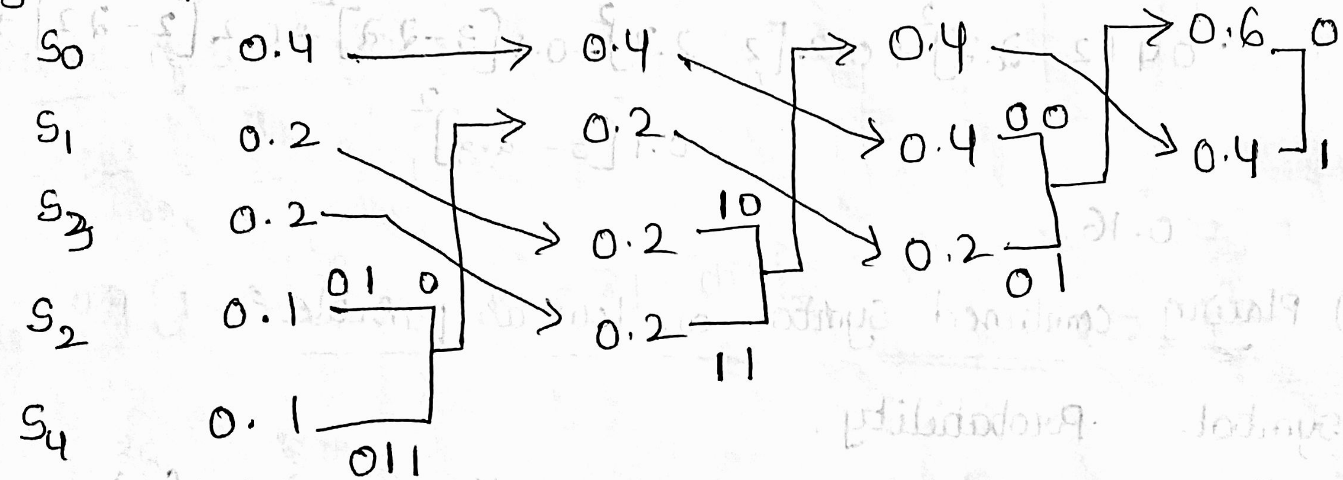
calulate the variance of the ensemble as defined by.

$$\sigma^2 = \sum_{K=0}^{M-1} P_K \left[I_K - I_{avg} \right]^2$$

solution :-

i) placing combined symbol as high as possible :-

Symbol probability



Symbol	probability	Codeword	length n_i
S_0	0.4	00	2
S_1	0.2	10	2
S_3	0.2	11	2
S_2	0.1	010	3
S_4	0.1	011	3

→ Average codeword length :-

$$N = \sum_{k=0}^{4} P_k n_k$$

$$= 0.4(2) + 0.2(2) + 0.1(3) + 0.2(2) + 0.1(3)$$

$$= 2.2 \text{ bits} / \text{symbol}.$$

→ variance of code.

$$\sigma^2 = \sum_{k=0}^{m-1} P_K \left[I_k - I_{avg} \right]^2$$

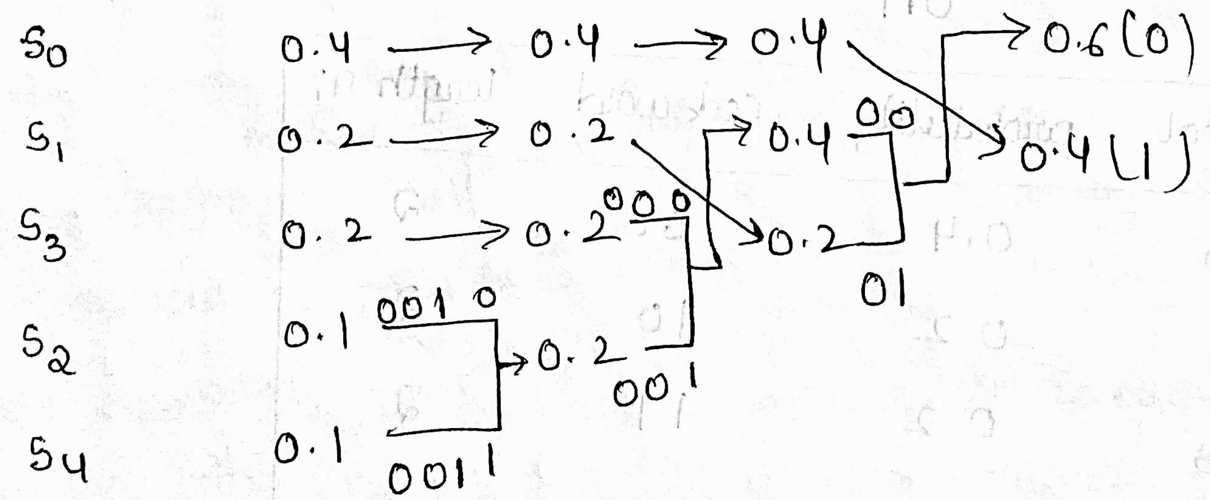
$$\sigma^2 = \sum_{k=0}^{4} P_K \left[n_K - \bar{N} \right]^2$$

$$= 0.4\left[2 - 2.2\right]^2 + 0.2\left[2 - 2.2\right]^2 + 0.1\left[3 - 2.2\right]^2 + 0.2\left[2 - 2.2\right]^2 + 0.1\left[3 - 2.2\right]^2$$

$$= 0.16.$$

ii) <u>Placing</u> <u>combined</u> <u>symbol</u> as low as <u>possible</u> :-

Symbol · Probability.



symbol	probability.	codeword	length n_i
S_0	0.4	1	1
S_1	0.2	01	2
S_3	0.2	000	3
S_2	0.1	0010	4
S_4	0.1	0011	4.

Average codeword length :-

$$N = 0.4(1) + 0.2(2) + 0.2(3) + 0.1(4) + 0.1(4)$$

$$= 2.2 \text{ bits / symbol.}$$

variance of the code :-

$$\sigma^2 = \sum_{K=0}^{4} P_K [n_K - N]^2$$

$$= 0.4[1-2.2]^2 + 0.2[2-2.2]^2 + 0.1[4-2.2]^2 + 0.2[3-2.2]^2 + 0.1[4-2.2]^2$$

$$= 1.36.$$

Result :-

method.	Average length	variance.
As high as possible	2.2	0.16.
As low as possible	2.2	1.36.

Above results show that average length of the codeword is same in both the methods. But minimum variance of huffman code is obtained by moving the probability of a combined symbol as high as possible.

Huffman Ternary coding :-

→ In binary coding, two symbols having lowest probabilities are combined. Huffman ternary coding combines three symbols having lowest probabilities. The combined probability of these symbols is placed at appropriate level in the next stage.

→ The Three symbols which are combined are assigned the codes 0, 1, and 2.

→ The remaining part is same as binary coding.

→ If there are 'M' symbols then.

$$M = r + (r-1)\alpha$$

Here for ternary coding $r = 3$ then.

$$M = 3 + 2\alpha \quad \text{or} \quad \alpha = \frac{M-3}{2}.$$

The value α should be an integer. ($\alpha = 1, 2, 3, \ldots$)

then $M = 5, 7, 9, 11 \ldots$

Then necessary number of symbols must be appended with zero probability. These are dummy symbols.

Huffman Quarternary coding :-

→ In Huffman quarternary coding combines four symbols having lowest probability in every stage. The combined probability is placed at appropriate level in the next stage.

→ The four symbols which are combined are assigned the codes 0, 1, 2, 3. remaining same as binary coding.

→ If There are 'M' symbols.

$M = r + (r-1)\alpha$. for quarternary coding $r = 4$.

$\therefore M = 4 + 3\alpha$ or $\alpha \equiv \frac{M-4}{3}$.

The value α should be an integer ($\alpha = 1, 2, 3 \ldots$)

then $M = 7, 10, 13, 16$.

→ Entropy of the source for ternary coding.

The entropy in r units per symbols is obtained as.

$$H_r = \frac{H}{\log_2 r} \qquad r = 3. \text{ for ternary code}$$

$$H_3 = \frac{H}{\log_2 3}.$$

→ Entropy of the source for quarternary coding.

The entropy in r units per symbols.

$$H_r = \frac{H}{\log_2 r} \qquad r = 4 \text{ for quarternary code.}$$

$$H_4 = \frac{H}{\log_2 4} = \frac{H}{2}.$$

1. Find the coding efficiency for the following messages using huffman Coding with $M = 3$.

x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$P(x_i)$	0.1	0.25	0.15	0.05	0.15	0.1	0.05	0.15.

Solution :- 1. Check number of symbols for huffman ternary coding.
2. obtain entropy of the binary source.
3. obtain entropy of the source for ternary coding.
4. obtain huffman ternary codes.
5. obtain average codeword length. 6) coding efficiency.

Step 1 :- check number of symbols for huffman ternary coding.

The length of the symbols are equal to

$$M = r + (r-1)\alpha$$

For ternary coding $r = 3$

$$M = 3 + (3-1)\alpha$$
$$= 3 + 2\alpha$$
$$= 5, 7, 9, 11 \ldots$$ for 'α' to be integer.

→ In the given alphabet there are '8' symbols. Hence we have to append the extra 9th symbol with zero probability.

x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$P(x_i)$	0.1	0.25	0.15	0.05	0.15	0.1	0.05	0.15	0

Step 2 :- Entropy :-

$$H = \sum_{K=1}^{9} P_K \log_2 \frac{1}{P_K}$$

$$= 0.1 \log_2\left(\frac{1}{0.1}\right) + 0.25 \log_2\left(\frac{1}{0.25}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right) + 0.05 \log_2\left(\frac{1}{0.05}\right) +$$
$$0.15 \log_2\left(\frac{1}{0.15}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right) + 0.05 \log_2\left(\frac{1}{0.05}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right) + 0$$

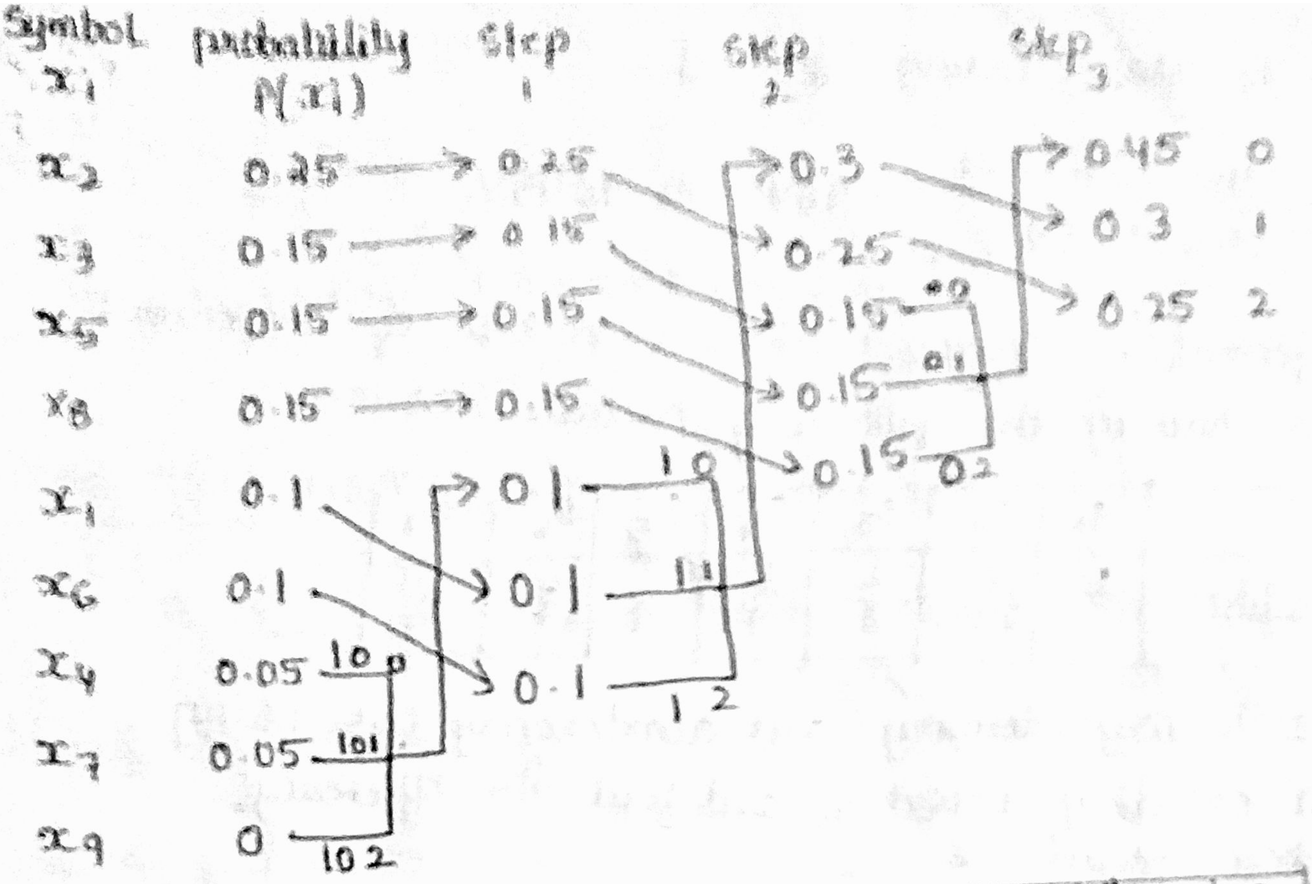
= 2.8282 bits / symbol.

Step 3 :- Entropy of the source for ternary coding :-

$$H_r = \frac{H}{\log_2 r} = \frac{2.8282}{\log_2 3} = 1.7844$$ ternary units / symbol.

Step 4 :- To obtain huffman ternary codes.

It combines last three symbols every time and assigns codes 0, 1 and 2. The symbols are arranged in the order of reducing probabilities.



symbol	probability	Huffman code	length of the code n_k
x_2	0.25	2	1
x_3	0.15	00	2
x_5	0.15	01	2
x_8	0.15	02	2
x_1	0.1	11	2
x_6	0.1	12	2
x_4	0.05	100	3
x_7	0.05	101	3
x_9	0	102	3

step 5 :- average code word length.

$$N_3 = \sum_{k=1}^{9} P_k n_k$$

$$= (0.25)(1) + 3\times(0.15\times2) + 2\,(0.15\times2) + 2\,(0.05\times3).$$

$$= 1.85 \text{ ternary units / symbol.}$$

Step 6 :- To obtain coding efficiency

$$\eta = \frac{H_3}{N_3} = \frac{1.7844}{1.85} = 0.9645 \text{ or } 96.45\%.$$

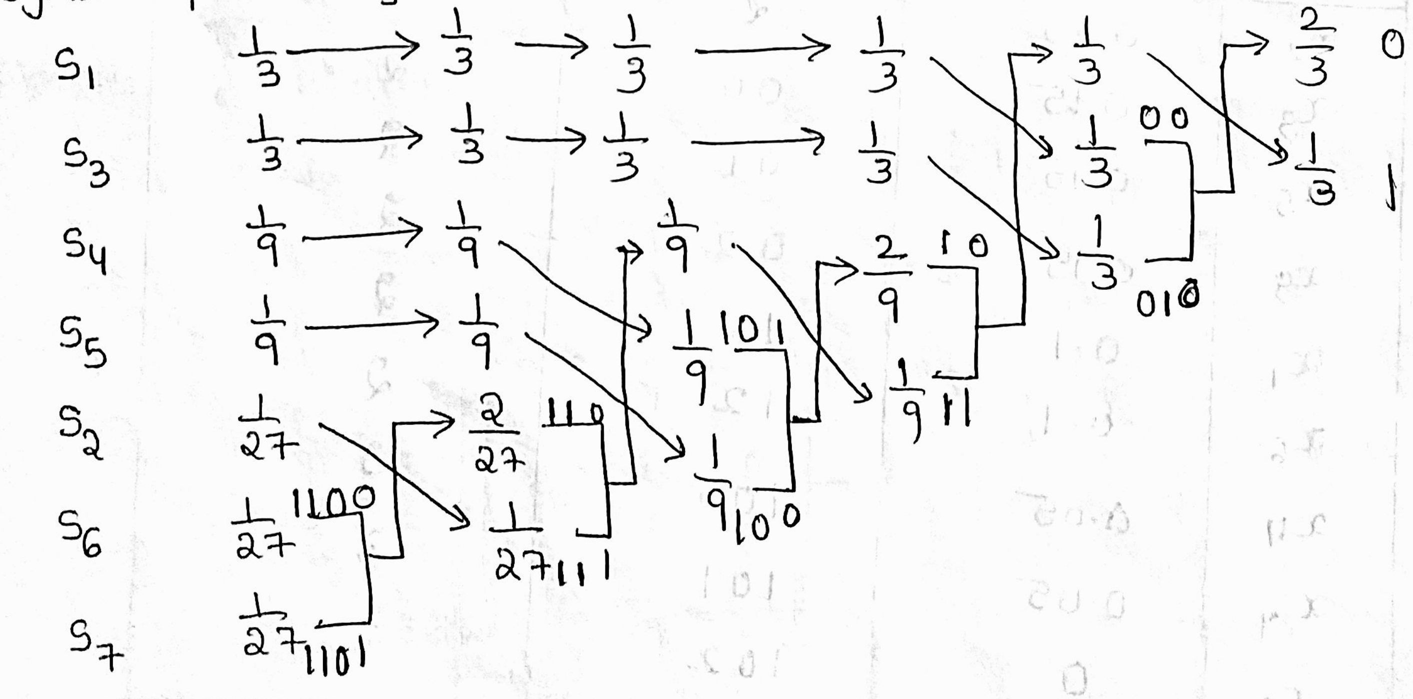
2. An information source produces a sequence of independent symbols having the following probabilities :-

Symbol	S_1	S_2	S_3	S_4	S_5	S_6	S_7
probabilities	$\frac{1}{3}$	$\frac{1}{27}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$.

Construct binary, ternary and quarternary code using huffman encoding procedure and find its efficiency. (As high as possible).

Solution :- 1) To obtain huffman binary codes.

Symbol probability.



Entropy :-

$$H = \sum_{k=1}^{7} P_k \log_2 \frac{1}{P_k}$$

$$= 2\times\frac{1}{3}\log_2(3) + 2\times\frac{1}{9}\log_2(9) + 3\times\frac{1}{27}\log_2(27)$$

$$= 2.2893903 \text{ bits} | \text{symbol}.$$

Average codeword length.

$$N = \sum_{k=1}^{7} p_k n_k$$

$$= \frac{1}{3}\times 2 + \frac{1}{3}\times 2 + \frac{1}{9}\times 3 + \frac{1}{9}\times 3 + \frac{1}{27}\times 3 + \frac{1}{27}\times 4 + \frac{1}{27}\times 4$$

$$= 2.4074 \text{ bits} / \text{symbol.}$$

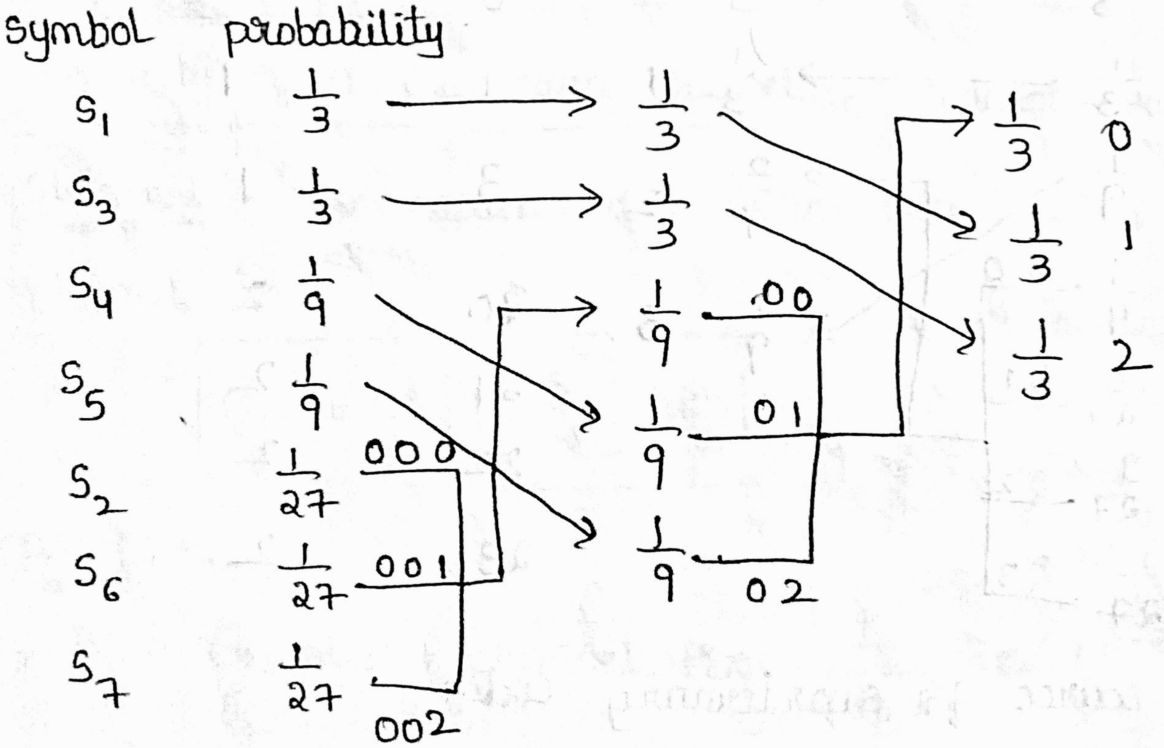
Code efficiency :-

$$\eta = \frac{H}{N} = \frac{2.2894}{2.4074} = 0.95098 \text{ or } 95.098\%.$$

ii) Huffman ternary coding :-

Since given number of symbols is 7. there is no need of appending any symbol to the source alphabet.

→ To obtain huffman ternary codes :-



→ Entropy of the source for ternary coding

$$H_r = \frac{H}{\log_2 r}$$

$$H_3 = \frac{2.2893}{\log_2 3} = 1.444 \text{ ternary units} / \text{symbol}$$

→ Average codeword length.

$$N = \sum_{K=1}^{7} P_K n_K$$

$$= \frac{1}{3}\times 1 + \frac{1}{3}\times 1 + \frac{1}{9}\times 2 + \frac{1}{9}\times 2 + \frac{1}{27}\times 3 + \frac{1}{27}\times 3 + \frac{1}{27}\times 3$$

= 1.4444. quarternary units | symbol.

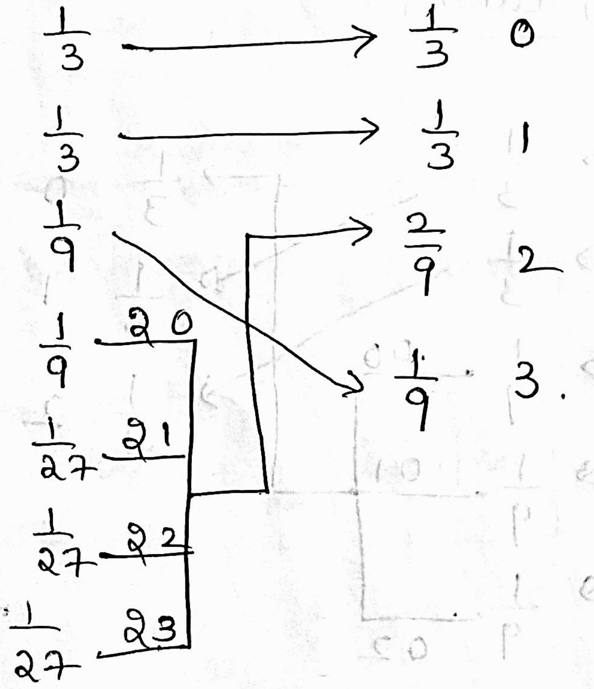
→ code efficiency

$$\eta = \frac{H_3}{N_3} = \frac{1.4444}{1.4444} = 1 \text{ or } 100\%.$$

iii) Huffman quarternary coding :-

Since given number of symbol is '7', there is no need of appending any symbol to the source alphabet.

→ To obtain huffman quarternary code.



Symbol	probabilities	Code	length n_K
s_1	$\frac{1}{3}$	0	1
s_3	$\frac{1}{3}$	1	1
s_4	$\frac{1}{9}$	3	1
s_5	$\frac{1}{9}$	20	2
s_2	$\frac{1}{27}$	21	2
s_6	$\frac{1}{27}$	22	2
s_7	$\frac{1}{27}$	23.	2.

→ Entropy of source for quarternary Coding.

$$H_4 = \frac{H}{\log_2 4} = \frac{2.2893}{2} = 1.1446 \text{ quarternary units} | \text{ symbol.}$$

→ average codeword length

$$N = \sum_{K=1}^{7} P_K n_K$$

$$= \frac{1}{3}\times 1 + \frac{1}{3}\times 1 + \frac{1}{9}\times 1 + \frac{1}{9}\times 2 + \frac{1}{27}\times 2 + \frac{1}{27}\times 2 + \frac{1}{27}\times 2$$

= 1.2222 quardernary units/symbol.

→ code efficiency.

$$\eta_4 = \frac{H_4}{N_4} = \frac{1.14469}{1.222} = 0.9365 \text{ or } 93.65\%.$$

Comments :-

Sr. No	Type of coding	Coding efficiency
1	Binary.	95.098%.
2	Ternary	100%.
3	quarternary.	93.65%.

* capacity of a Gaussion channel :- (Shannon - Hartley theorem).

Statement :- The capacity of a white - band limited gaussian channel is

$$\boxed{C = w \log\left[1 + \frac{S}{N}\right] \text{ bits/sec}}$$

w = channel Bandwidth.
S = average signal power
N = average noise power.

proof :-

For a Gaussian channel, $P(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \longrightarrow ①$

The entropy can be $H(x) = -\int_{-\infty}^{\infty} P(x) \log_2 P(x)\, dx \longrightarrow ②$

Apply logarithm on both sides to ①

$$\log P(x) = \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \bar{e}^{x^2/2\sigma^2} \right]$$

$$= \log \frac{1}{\sqrt{2\pi\sigma^2}} + \log \bar{e}^{x^2/2\sigma^2}$$

$$= -\log \sqrt{2\pi\sigma^2} - \log e^{x^2/2\sigma^2}$$

$$-\log P(x) = \log \sqrt{2\pi\sigma^2} + \log e^{x^2/2\sigma^2}$$

Substituting the above equation in ②

$$H(x) = \int_{-\infty}^{\infty} P(x) \log \sqrt{2\pi\sigma^2}\, dx + \int_{-\infty}^{\infty} P(x) \log \bar{e}^{x^2/2\sigma^2}\, dx$$

The above integral can be evaluated as.

$$H(x) = \log \sqrt{2\pi e\sigma^2}$$

If the signal is band limited to B Hz, then the rate of information can be given as.

$$R = rH \qquad r = fs$$

$$fs = 2B$$

$$R(x) = rH$$

$$= 2B \log \sqrt{2\pi e\sigma^2}$$

$$= B \log \left[\left(2\pi e\sigma^2 \right)^{1/2} \right]^2$$

$$= B \log 2\pi e N \quad \left[\sigma^2 = N \right]$$

Consider a continuous source transmitting information over noisey channel. If the received signal is composed of a transmitted signal x & a noise n, then the joint entropy of source & noise is given by

$\therefore R(X,n) = R(X) + R(n/X)$

Assuming that the transmitted signal & noise are independent, then

$R(X,n) = R(X) + R(n)$

Since the received signal Y is the sum of the transmitted signal X and the noise n

$H(x,Y) = H(x,n)$

$H(Y) + H(X/Y) = H(X) + H(n)$.

(or) $R(Y) + R(X/Y) = R(X) + R(n) \rightarrow$ ③

The rate at which information is received from a noisy channel is $R = R(X) - R(X/Y)$.

from equation ③, the above equation can be written as

$R = R(Y) - R(n)$ bits/sec.

The channel capacity in bit/sec is

$C = \max [(R)]$ bits/sec

$C = \max [R(Y) - R(n)]$

$C = \max [w \log (2\pi e (S+N)) - w \log (2\pi e N)]$

$C = w \log \left(\frac{S+N}{N}\right)$

$(\log ~~(A - \log B)~~ \Rightarrow$

$\log A - \log B = \log \frac{A}{B}$.

$$\boxed{C = w \log \left(1 + \frac{S}{N}\right)}$$

Bandwidth and S/N trade-off :-

channel capacity of gaussian channel is given by

$$C = \omega \log_2 \left(1 + \frac{S}{N}\right)$$

For a noiseless channel, signal noise is zero

$$C = \omega \log_2 \left(1 + \frac{S}{0}\right) = \omega \log_2 (1 + \infty) = \infty$$

Noiseless channel has infinite bandwidth.

For a fixed signal power & in the presence of white gaussian noise, the channel capacity approaches an upper limit (known as 'shannon limit') with bandwidth increased to infinity.

$$C = \omega \log \left[1 + \frac{S}{N}\right)$$

$$\Rightarrow C = \omega \log \left(1 + \frac{S}{n\omega}\right)$$

$$C = \frac{S}{n} \frac{n\omega}{S} \log \left(1 + \frac{S}{n\omega}\right)$$

$$= \frac{S}{n} \log \left(1 + \frac{S}{n\omega}\right)^{n\omega/S}$$

If $x = \frac{S}{n\omega}$ approaches ∞, x approaches zero.

$$\left(\because \lim_{x \to 0} (1+x)^{1/x} = e \right.$$

$$\lim_{\omega \to \infty} \left(1 + \frac{S}{n\omega}\right)^{n\omega/S} = e$$

$$\lim_{\omega \to \infty} C = \frac{S}{n} \log e = 1.44 \frac{S}{n} = R_{max}$$

Consider the trade-off between bandwidth & S/N ratio.

let $\frac{S}{N} = 15$ & $W = 5KHz$ then

$$C = w \log\left(1+\frac{S}{N}\right)$$

$$= 5 \log(1+15) = 20K \text{ bits/sec.}$$

If $\frac{S}{N}$ ratio is increased to 31, the bandwidth for the same channel capacity is

$$C = 20$$

$$C = w \log(1+31)$$

$$w = \frac{20}{\log 32} = \frac{20}{5} = 4KHz.$$

$\frac{S}{N}\uparrow$ B.W$\downarrow$

B.W$\downarrow$ S.P$\uparrow$

B.W$\uparrow$ S.P$\downarrow$

So with the same channel capacity as $\frac{S}{N}$ is increased the band width reduces. So to decrease band width, the signal power has to be increased & to decrease the signal power, the band-width must be increased.

problems :-

1. A Gaussian channel has 1MHz bandwidth. calculate the channel capacity. if the signal power to noise spectral density ratio (S/N) is 10^5 Hz. Also find the maximum information rate.

$$C = w \log\left(1+\frac{S}{N}\right)$$

$$= w \log\left(1+\frac{S}{\eta w}\right)$$

$$= 10^6 \log\left(1+\frac{10^5}{10^6}\right) = 13,800 \text{ bits/sec}$$

maximum information rate = $R_{max} = 1.44 \frac{S}{\eta}$

$= 1.44 \times 10^5 \Rightarrow 144,000$ bits/sec)

Channel capacity of discrete and analog channels :-

The channel capacity theorem for bandlimited powerlimited white Gaussian Noise channels.

step1 :- let us assume a zero mean stationary process $x(t)$ is bandlimited to BHZ. let X_K $K=1,2,\ldots n$ indicates the continuous random variables obtained by sampling $x(t)$. let Y_K $k=1,2,\ldots n$ denote the samples of received signal. They are related to X_K as,

$$Y_K = X_K + N_K$$

Here N_K are samples of white Gaussian Noise of zero mean and variance of σ^2.

step2 :- The channel capacity for the channel described in above step

$$C = \max_{f_{X_K}} (x) \{ I(X_K ; Y_K) : X_K \text{ Gaussian} \}$$

Here $I(x_K : Y_K)$ is average mutual information and $f_{X_K}(x)$ is pdf of X_K.

step3 :- The average mutual information $I(X_K : Y_K)$ is given as

$$I(X_K ; Y_K) = h(Y_K) - h\left(\frac{Y_K}{X_K}\right) \qquad h\left(\frac{Y_K}{X_K}\right) = h(N_K)$$

$$I(X_K : Y_K) = h(Y_K) - h(N_K)$$

Step4 :- The variance of Y_K equals $S+\sigma^2$. Here S is the average transmitted power. Hence from equation.

$$h(Y_K) = \frac{1}{2}\log_2 (2\pi e(S+\sigma^2)] \text{ and}$$

$$h(N_K) = \frac{1}{2}\log_2 (2\pi e\sigma^2).$$

Step5 :- $I(X_K ; Y_K) = \frac{1}{2}\log_2 [2\pi e(S+\sigma^2)] - \frac{1}{2}\log_2 (2\pi e\sigma^2)$

$$= \frac{1}{2}\log_2 \left(1+\frac{S}{\sigma^2}\right).$$

∴ Thus is nothing but the channel capacity of Gaussian channel

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{\sigma^2}\right)$$

when this channel is used over the bandwidth of B HZ, then above equation becomes,

$$C = B \log_2 \left[1 + \frac{S}{N}\right) \text{ bits/sec}$$

problems :-

1. Calculate the channel capacity for AWGN channel with band width of 1 MHZ and signal to noise ratio of -40db.
2. Show that the channel capacity of an ideal AWGN channel with infinite bandwidth is by $C_\infty = 1.44 \frac{S}{N}$ b/s. where S is the average signal power $\eta/2$ is the power spectral density of the Gaussian Noise
3. Explain about Shannon - Hartley law ? And find the channel capacity of a AWGN channel with 4KHZ bandwidth and noise power spectral density $\eta/2 = 10^{-12}$ W/HZ. The signal power required at the receiver is 0.1 mW.
4. Explain about Bandwidth - S/N trade off.

ii) Convolution codes :- The coding operation is discrete time convolution of input sequence with the impulse response of the encoder. The convolution encoder accepts the message bits continuously and generate the encoded sequence continuously.

The codes can also be classified as linear & non-linear codes

i) linear code:- If the two codewords of the linear code are added by modulo-2 arithmetic, then it produces third codeword in the code.

ii) non-linear code:- Addition of the nonlinear codewords does not necessarily produce third codeword.

> methods of controlling errors :-

There are two main methods used for error control coding
1. Forward acting error correction.
2. Error detection with Transmission.

1) Forward acting error correction :-

In this method, The check bits or redundant bits are used by the receiver to detect and correct errors. The error correction and detection capability of the receiver depends upon number of redundant bits in the transmitted message. (it is ~~fast~~, probability of errors is higher)

| Error detection with retransmission

In this method, the decoder checks the input sequence. when it detects any error, it discards that part of the sequence and requests the transmission for retransmission. (it is slow, probability of errors is lower).

of Errors :-

There are mainly two types of errors introduced during transmission on the data: 1. random errors 2. Burst errors.

i) **Random errors** :- These errors are created due to white gaussian noise in the channel. The errors generated due to white gaussian noise in the particular interval does not affec the performance of the system. Hence they are called as random errors.

ii) **Burst errors** :- These errors are generated due to impulsive noise in the channel. These impulse noise are generated due to lightning and switching transients. These noise bursts affects several successive symbols. such errors are called burst errors.

* Some of the important Terms used in Error control coding :-

The terms which are regularly used in error control coding.

→ **Code word** :-

The encoded block of 'n' bits is called a codeword. If contains message bits and redundant bits.

→ **Block length** :-

The number of bits 'n' after coding is called the block length of the code.

→ **Code rate** :-

The ratio of message bits (K) and the encoder output bits (n) is called code rate. code rate is defined by r

$$\boxed{r = \frac{k}{n}}$$

we find that $0 < r < 1$

channel data rate :-

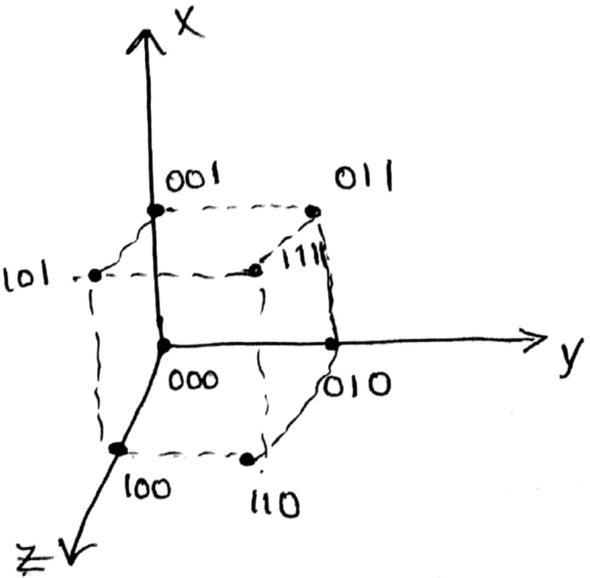
It is the bit rate at the output of encoder. If bit rate at the input of encoder is R_s, then channel data rate will be.

$$\boxed{\text{channel data rate } (R_o) = \frac{n}{k} R_s}$$

code vectors :-

An 'n' bit codeword can be visualized in an n- dimension space as a vector whose elements or co-ordinates are the bits in the codeword. It is simpler to visualize the 3-bit codewords. There will be distinct 8 codewords (since number of codewords = 2^k). If we let bits b_0 on x-axis, b_1 on y-axi and b_2 on z-axis.

Sr. No	Bits of code vector		
	$b_2 = z$	$b_1 = y$	$b_0 = x$
1	0	0	0
2	0	0	1
3	0	1	0
4	0	1	1
5	1	0	0
6	1	0	1
7	1	1	0
8	1	1	1



code vectors representing 3-bit code words.

code vectors in 3- dimensional space.

Hamming distance :-

The hamming distance between the two code vectors is equal to the number of elements in which they diffe

ample let $X = (101)$ and $Y = 110$. The two code vectors
r in second and third bits.

$$d(X, Y) = d = 2.$$

minimum distance (dmin) :- It is smallest hamming distance between the valid code vectors.

Sr. NO	Name of errors detected/corrected	Distance requirement
1.	Detect upto 's' errors per word	$d_{min} \geq S+1$.
2.	Correct upto 't' errors per word	$d_{min} \geq 2t+1$
3.	Correct upto 't' errors and detect S>t errors per word.	$d_{min} \geq t+S+1$.

For the (n, K) block code, the minimum distance is given as

$$d_{min} \leq n-k+1.$$

Code efficiency :- The code efficiency is the ratio of message bits in a block to the transmitted bits for that block by the encoder.

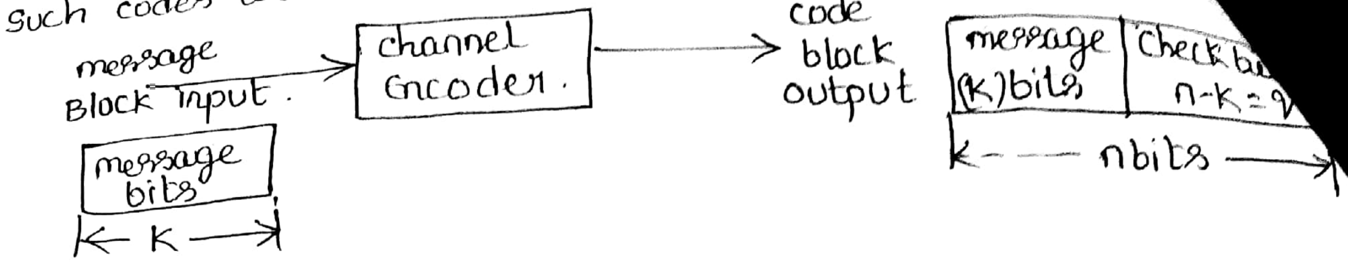
$$\text{code efficiency} = \frac{\text{message bits in a block}}{\text{transmitted bits for the block}} = \frac{K}{n}.$$

weight of the code :- The number of non-zero elements in the transmitted code vector is called vector weight. It is denoted by $w(X)$ where x is the code vector. $X = 1010110$, then weight of this code vector will be $w(X) = 4$.

linear block codes :-

principle of block coding :- For the block of K message bits, (n-K) parity bits or check bits are added. Hence the total bits at the output of channel encoder are 'n'.

Such codes are called (n, K) block codes.



Systematic codes:- In the systematic block code, the message bits appear at the begiming of the code word and then check bits are transmitted in a block. This type of code is called systematic code.

Non-systematic codes:- In non-systematic code it is not possible to identify message bits and check bits. They are mixed in the block.

linear code:- A code is linear if the sum of any two code vectors produces another code vector.

consider that the particular code vector consists of $m_1, m_2, m_3 \ldots m_k$ message bits and $c_1, c_2, c_3 \ldots, c_q$ check bits. Then this code vector can be written as.

$$X = (m_1, m_2, \ldots, m_k\ c_1, c_2 \ldots c_q) \quad \text{Here } q = n-k.$$

The above code vector can also be written as.

$$X = (M|C)$$

Matrix description of linear Block codes:-

The code vector can be repeiresented as.

$$X = MG.$$

X : code vector of 1xn size or n bi

M = message vector of 1xK size

G = Generator matrix of Kxn size

then

$$[X]_{1\times n} = [M]_{1\times K} \cdot [G]_{K\times n}.$$

generator matrix depends upon the linear block code sed.

$$G = [I_K \mid P_{K \times q}]_{K \times n}$$

I_K = K×K Identity matrix &

P = K×q Submatrix.

The check vector can be obtained as $C = MP$.

$$[c_1, c_2, \ldots, c_q]_{1 \times q} = [m_1, m_2, \ldots m_K]_{1 \times K} \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1q} \\ P_{21} & P_{22} & \cdots & P_{2q} \\ \vdots \\ P_{K1} & P_{K2} & \cdots & P_{Kq} \end{bmatrix}_{K \times q}$$

check vector can be obtained.

$$c_1 = m_1 P_{11} \oplus m_2 P_{21} \oplus m_3 P_{31} \oplus \cdots \oplus m_K P_{K1}$$
$$c_2 = m_1 P_{12} \oplus m_2 P_{22} \oplus m_3 P_{32} \oplus \cdots \oplus m_K P_{K2}$$
$$c_3 = m_1 P_{13} \oplus m_2 P_{23} \oplus m_3 P_{33} \oplus \cdots \oplus m_K P_{K3}.$$

1) The generator matrix for a (6,3) block code is given below. Find all code vectors of this code.

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Solution :- (6,3) block code. In this code $n = 6$, $K = 3$,

$$q = n - K = \underline{3}$$

i) To obtain P sub matrix :-

$$G = [I_K \vdots P_{K \times q}]$$

$$I_K = I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$P_{K \times q} = P_{3 \times 3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

ii) To obtain the equations for check bits :-

$$[c_1\, c_2\, c_3] = [m_1\, m_2\, m_3] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$c_1 = (0 \times m_1) \oplus m_2 \oplus m_3 = m_2 \oplus m_3$$

$$c_2 = m_1 \oplus (0 \times m_2) \oplus m_3 = m_1 \oplus m_3$$

$$c_3 = m_1 \oplus m_2 \oplus (0 \times m_3) = m_1 \oplus m_2$$

iii) To determine check bits and code vectors for every message vector

consider the first block of $(m_1 m_2, m_3) = 000$ we have.

$c_1 = 0 \oplus 0 = 0$, $c_2 = 0 \oplus 0 = 0$, $c_3 = 0 \oplus 0 = 0$.

then $c_1 c_2 c_3 = 000$. similarly for remaining code vectors

Sr. No	message bits.			check bits			code vector						weight of Code $w(\bar{X})$
	m_1	m_2	m_3	c_1	c_2	c_3	m_1	m_2	m_3	c_1	c_2	c_3	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	1	1	1	0	3
3	0	1	0	1	0	1	0	1	0	1	0	1	3
4	0	1	1	0	1	1	0	1	1	0	1	1	4
5	1	0	0	0	1	1	1	0	0	0	1	1	3
6	1	0	1	1	0	1	1	0	1	1	0	1	4
7	1	1	0	1	1	0	1	1	0	1	1	0	4
8	1	1	1	0	0	0	1	1	1	0	0	0	3

Parity check matrix (H)

For every block code there is a $q \times n$ parity check matrix (H). It is defined by

$$H = [P^T : I_q]_{q \times n}.$$

Here P^T is the Transpose of P- submatrix.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1q} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2q} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3q} \\ \vdots \\ P_{K1} & P_{K2} & P_{K3} & \cdots & P_{Kq} \end{bmatrix}_{K \times q} \quad \text{then } P^T = \begin{bmatrix} P_{11} & P_{21} & P_{31} & \cdots & P_{K1} \\ P_{12} & P_{22} & P_{32} & \cdots & P_{K2} \\ P_{13} & P_{23} & P_{33} & \cdots & P_{K3} \\ \vdots \\ P_{1q} & P_{2q} & P_{3q} & \cdots & P_{Kq} \end{bmatrix}$$

$$H_{q \times n} = \left[\begin{array}{ccccc|ccccc} P_{11} & P_{21} & P_{31} & \cdots & P_{K1} & 1 & 0 & 0 & \cdots & 0 \\ P_{12} & P_{22} & P_{32} & \cdots & P_{K2} & 0 & 1 & 0 & \cdots & 0 \\ P_{13} & P_{23} & P_{33} & \cdots & P_{K3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ P_{1q} & P_{2q} & P_{3q} & \cdots & P_{kq} & 0 & 0 & 0 & & 1 \end{array}\right]_{q \times n}.$$

Hamming codes :-

Hamming codes are (n, K) linear block codes. These codes satisfy the following conditions.

1. Number of check bits $q \geq 3$.
2. Block length $n = 2^q - 1$
3. number of message bits $k = n - q$.
4. minimum distance $d_{min} = 3$.

Code rate $r = \frac{k}{n} = \frac{n-q}{n} = 1 - \frac{q}{n}$.

putting the value of $n = 2^q - 1$ we get,

$$r = 1 - \frac{q}{2^q - 1} \quad (r = 1 \text{ if } q >> 1).$$

Error detection and correction capabilities of Hamming codes :-

The minimum distance of Hamming code is 3.

For detecting s errors per word then $d_{min} \geq s + 1$

$$3 \geq s + 1$$
$$s \leq 2$$

For correcting t errors per word then

$$d_{min} \geq 2t + 1$$

$$3 \geq 2t + 1 \Rightarrow t \leq 1$$

It can be used to detect double errors or correct single errors.

③ The parity check matrix of a particular (7,4) linear bloc
is given by

$$H = \begin{bmatrix} 1110 & 100 \\ 1101 & 010 \\ 1011 & 001 \end{bmatrix}$$

i) find the G.

ii) list all the code vectors.

iii) what is the minimum distance between code vectors.

iv) how many errors can be detected? How many errors can be corrected?

Solution :-

Here $n = 7$ and $k = 4$.

$q = n - k = 3$.

i) To obtain the Generator matrix (G) :-

$$H = [P^T \vdots I] \qquad G = [I_k \vdots P_{k \times q}]_{k \times n}$$

$$H = \begin{bmatrix} 1110 & 100 \\ 1101 & 010 \\ 1011 & 001 \end{bmatrix} \qquad G = \underbrace{\begin{bmatrix} 1000 & 111 \\ 0100 & 110 \\ 0010 & 101 \\ 0001 & 011 \end{bmatrix}}_{k \times n \quad 4 \times 7}$$

$$\underbrace{\quad}_{I_{4 \times 4}} \quad \underbrace{\quad}_{P_{4 \times 3}}$$

then

$$P^T = \begin{bmatrix} 1110 \\ 1101 \\ 1011 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

ii) To find all the code words :-

$$C = MP$$

$$[C_1 \, C_2 \, C_3]_{1 \times 3} = [m_1 m_2 \, m_3 \, m_4] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{4 \times 3}$$

$= m_1 \times 1 \oplus 1 \times m_2 \oplus 1 \times m_3 \oplus 0 \times m_4 = m_1 \oplus m_2 \oplus m_3$

$C_2 = m_1 \times 1 \oplus m_2 \times 1 \oplus 0 \times m_3 \oplus 1 \times m_4 = m_1 \oplus m_2 \oplus m_4.$

$C_3 = m_1 \times 1 \oplus m_2 \times 0 \oplus m_3 \times 1 \oplus 1 \times m_4 = m_1 \oplus m_3 \oplus m_4$

consider $m_1 m_2 m_3 m_4 = 1010$ then.

$C_1 = 1 \oplus 0 \oplus 1 = 0$

$C_2 = 1 \oplus 0 \oplus 0 = 1$

$C_3 = 1 \oplus 1 \oplus 0 = 0$

SR NO	message vector. $m_1\ m_2\ m_3\ m_4$	Check bits. $C_1\ C_2\ C_3$	code vector (or) codeword $m_1\ m_2\ m_3\ m_4\ C_1\ C_2\ C_3$	weight of code vector $w(X)$
1	0 0 0 0	0 0 0	0 0 0 0 0 0 0	0
2	0 0 0 1	0 1 1	0 0 0 1 0 1 1	3
3	0 0 1 0	1 0 1	0 0 1 0 1 0 1	3
4	0 0 1 1	1 1 0	0 0 1 1 1 1 0	4
5	0 1 0 0	1 1 0	0 1 0 0 1 1 0	3
6	0 1 0 1	1 0 1	0 1 0 1 1 0 1	4
7	0 1 1 0	0 1 1	0 1 1 0 0 1 1	4
8	0 1 1 1	0 0 0	0 1 1 1 0 0 0	3
9	1 0 0 0	1 1 1	1 0 0 0 1 1 1	4
10	1 0 0 1	1 0 0	1 0 0 1 1 0 0	3
11	1 0 1 0	0 1 0	1 0 1 0 0 1 0	3
12	1 0 1 1	0 0 1	1 0 1 1 0 0 1	4
13	1 1 0 0	0 0 1	1 1 0 0 0 0 1	3
14	1 1 0 1	0 1 0	1 1 0 1 0 1 0	4
15	1 1 1 0	1 0 0	1 1 1 0 1 0 0	4
16	1 1 1 1	1 1 1	1 1 1 1 1 1 1	7

iii) minimum distance between code vectors :-

The Smallest weight of any non-zero code vector is

then minimum distance is $d_{min} = 3$.

iv) Error detection and correction capabilities.

since $d_{min} = 3$

$d_{min} \geq s+1$

$3 \geq s+1$

$s \leq 2$

Thus two errors will be detected.

$d_{min} \geq 2t+1$

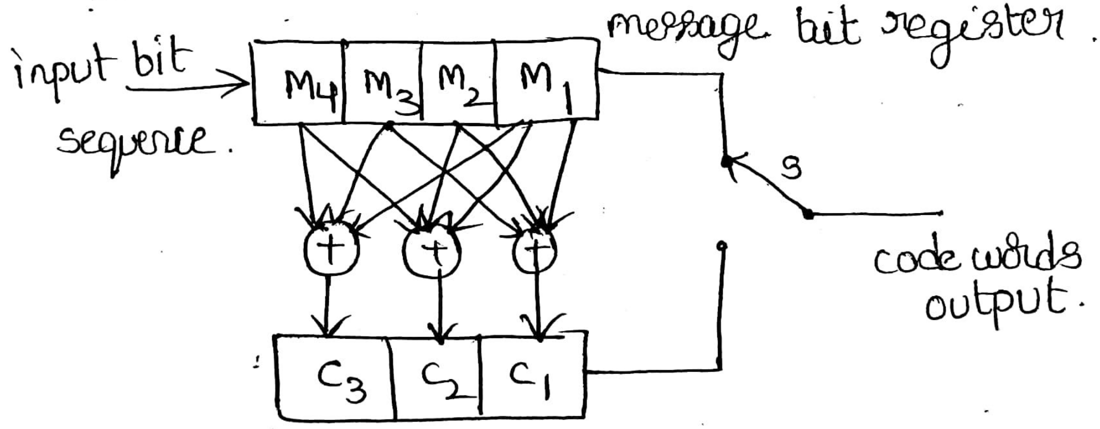
$3 \geq 2t+1$

$t \leq 1$

Thus one error will be corrected.

The hamming code $d_{min} = 3$ always two errors can be detected and single error can be corrected.

Encoder of (7,4) Hamming code :-



check bits register.

Encoder for (7,4) hamming code or
(7,4) linear block code.

The lower register contains check bits C_1, C_2 and C_3. These bits are obtained from the message bits by mod-2 addition These additions are performed according to equations. The mod-2 addition operation is nothing but EX-OR operation.

switch 's' is connected to message register first and (E l message bits are transmitted. The switch is then connected to the check bit register and check bits are transmitted. This forms a block of 7 bits.

Syndrome Decoding:-

Syndrome:-

Transmitted code vector be 'x' and corresponding received code vector be represented by 'y'. Then we can write.

$X = Y$ if there are no transmission errors.

$X \neq Y$ if there are errors created during transmission.

The non-zero output of the product YH^T is called syndrome and it is used to detect the errors in Y. Syndrome is represented by 's' and can be written as,

$$S = YH^T.$$

$$[S]_{1\times q} = [Y]_{1\times n} \left[H^T\right]_{n\times q}$$

$$H = \left[P^T : I_q\right]_{q\times n}.$$

The transpose of the above matrix can be obtained by interchanging the rows and the columns.

$$H^T = \begin{bmatrix} P \\ --- \\ I_q \end{bmatrix}_{n\times q}.$$

Important property used in syndrome decoding:-

$$X.H^T = (0\ 0\ 0 \cdots 0) \quad (\text{or}) \quad [X]_{1\times n}\left[H^T\right]_{n\times q} = (0\,0\,0\,0)_{1\times q}$$

This is true for all code vectors.

Detecting error with the help of syndrome and error

The non-zero elements of 's' represent error in the outp when all the elements of s are zero, the two cases are possible.

i) No error in the output and $Y = X$.

ii) Y is some other valid code vector other than X. This means the transmission errors are undetectable.

Ex:- $X = (101101)$ Transmitted vector.

$Y = (101001)$ received vector (with error).

$E = (000100)$ Error vector.

then $X = Y \oplus E$ (or) $Y = X \oplus E$.

$= (101001) \oplus (000100)$

$= (1\oplus0\ 0\oplus0\ 1\oplus0\ 0\oplus1\ 0\oplus0\ 1\oplus0)$

$= (1\ 01101)$

Relationship between syndrome vector (S) and error vector (E).

$S = YH^T$ $\quad\quad Y = X \oplus E$

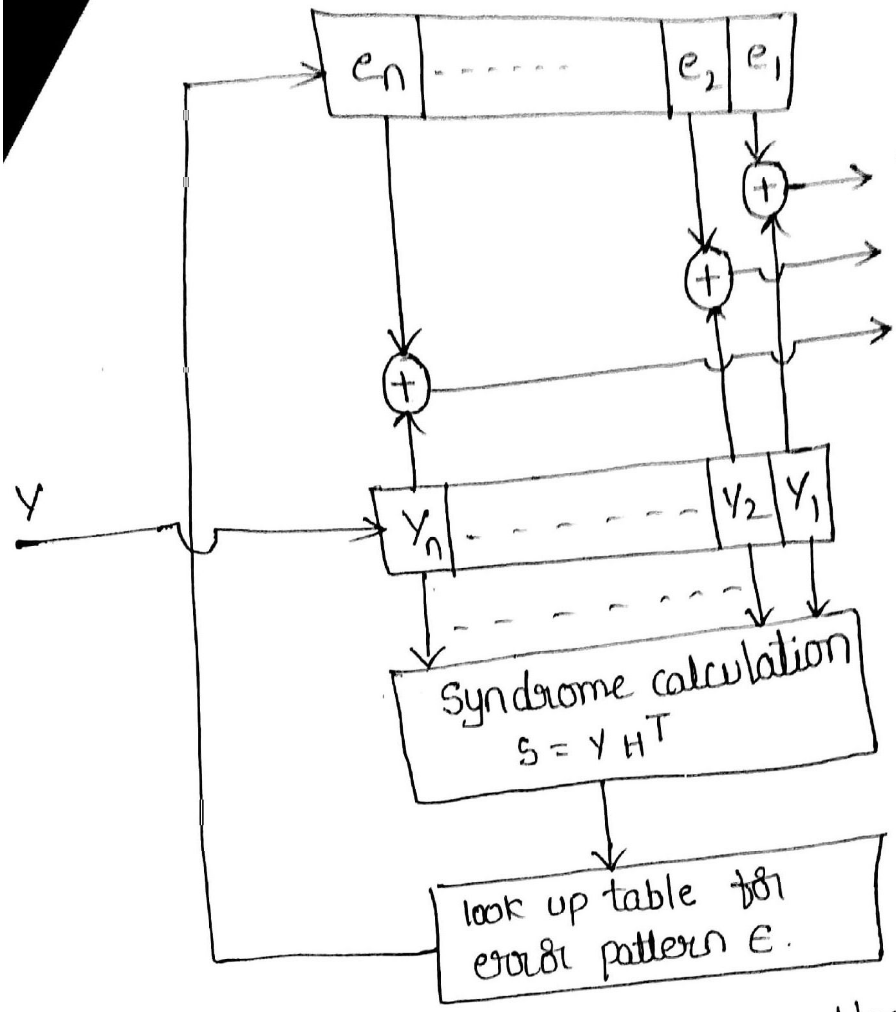
$S = (X \oplus E)H^T$

$S = XH^T \oplus EH^T$ $\quad\quad [X.H^T = 0]$

$S = EH^T$

This relation shows that syndrome depends upon the error pattern only. It does not depend upon a particular message.

ome decoder for (n,k) Block code:-



Syndrome decoder for linear block code:-

Syndrome decoder for linear block code to correct errors. The received n-bit vector 'Y' is stored in an n-bit register. From this vector a syndrome is calculated using.

$$S = Y . H^T .$$

Thus H^T is stored in the syndrome calculator. The q-bit syndrome vector is then applied to a look up table of error patterns. Depending upon the particular syndrome an error pattern is selected. This error pattern is added to the vector Y. The output is $X = Y \oplus E$. The Block diagram shows above can correct only single error.

problems on linear Block codes :-

3. The parity check bits of a (8,4) Block code are gene
by $c_5 = d_1 + d_2 + d_4$

$c_6 = d_1 + d_2 + d_3$

$c_7 = d_1 + d_3 + d_4$

$c_8 = d_2 + d_3 + d_4$

where d_1, d_2, d_3 and d_4 are the message digits.

i) find the generator matrix and parity check matrix for this code

ii) find the minimum weight of this code

iii) Find the error detecting and correcting capabilities of this code.

Solution:-

i) (8,4) Block code means

$n = 8, \; K = 4$ (message bits.

$q = n - K = 8 - 4$

$= 4$ (check bits)

$$C = M \cdot P_{K \times q}$$

$$C_5 C_6 C_7 C_8 = D_1 D_2 D_3 D_4 \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

$$C_5 = D_1 P_{11} \oplus D_2 P_{21} \oplus D_3 P_{31} \oplus D_4 P_{41}$$

$$C_6 = D_1 P_{12} \oplus D_2 P_{22} \oplus D_3 P_{32} \oplus D_4 P_{42}$$

$$C_7 = D_1 P_{13} \oplus D_2 P_{23} \oplus D_3 P_{33} \oplus D_4 P_{43}$$

$$C_8 = D_1 P_{14} \oplus D_2 P_{24} \oplus D_3 P_{34} \oplus D_4 P_{44}.$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Generator Matrix $(G) = [I_K : P_{K \times q}]_{K \times n}$

$$G = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

ii) To obtain the minimum weight of the code

$d_{min} = 3.$

iii) To obtain the error detecting and correcting capabilities of this code.

error detecting capabilities = $d_{min} \geq s+1$

$3 \geq s+1$

$s \leq 2.$

error correcting capabilities = $d_{min} \geq 2t+1$

$3 \geq 2t+1$

$t \leq 1.$

4. For a systematic linear block code, the three parity check bits c_4, c_5 and c_6 are given by.

$c_4 = m_1 \oplus m_2 \oplus m_3.$

$c_5 = m_1 \oplus m_2$

$c_6 = m_1 \oplus m_3.$

i) construct generator matrix.

ii) construct code generated by this matrix.

iii) determine error correcting capability.

iv) prepare a suitable decoding table.

v) Decode the received words 101100 and 000110.

Solution :-

i) To obtain Generator matrix:-

$$C = M \cdot P$$

$$C_4 C_5 C_6 = [m_1 \; m_2 \; m_3] \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$C_4 = m_1 P_{11} \oplus m_2 P_{21} \oplus m_3 P_{31}$ $\quad C_4 = m_1 \oplus m_2 \oplus m_3$

$C_5 = m_1 P_{12} \oplus m_2 P_{22} \oplus m_3 P_{32}$ $\quad C_5 = m_1 \oplus m_2$

$C_6 = m_1 P_{13} \oplus m_2 P_{23} \oplus m_3 P_{33}.$ $\quad C_6 = m_1 \oplus m_3.$

Compare the equation with given equations for C_4, C_5, C_6.

$P_{11} = 1, \; P_{12} = 1 \quad P_{13} = 1$

$P_{21} = 1 \quad P_{22} = 1 \quad P_{23} = 0$

$P_{31} = 1 \quad P_{32} = 0 \quad P_{33} = 1$

Hence, the parity matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3 \times 3}.$

$$G = [I_K : P] = [I_3 : P_{3 \times 3}]$$

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

ii) To obtain the codewords :- $C_4 = m_1 \oplus m_2 \oplus m_3$

$C_5 = m_1 \oplus m_2$

$C_6 = m_1 \oplus m_3$

	message vector			check bits			code vector (or) code words						code weight
	m_1	m_2	m_3	c_4	c_5	c_6	m_1	m_2	m_3	c_4	c_5	c_6	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	1	0	0	1	1	0	1	3
3	0	1	0	1	1	0	0	1	0	1	1	0	3
4	0	1	1	0	1	1	0	1	1	0	1	1	4
5	1	0	0	1	1	1	1	0	0	1	1	1	4
6.	1	0	1	0	1	0	1	0	1	0	1	0	3
7.	1	1	0	0	0	1	1	1	0	0	0	1	3
8	1	1	1	1	0	0	1	1	1	1	0	0	4.

iii) The error correcting capacity depends on the minimum distance d_{min}. $d_{min} = 3$.

Therefore, number of errors detectable is $d_{min} \geq S+1$

$$3 \geq S+1 \quad \text{or} \quad S \leq 2$$

Two errors can be detected.

and $d_{min} \geq 2t+1$

$$3 \geq 2t+1$$

$t \leq 1$. (one error can be corrected).

iv) To obtain decoding table :-

To write the decoding table, we have to calculate the syndrome S.

$$S = Y \cdot H^T$$

$$H^T = \begin{bmatrix} P \\ -- \\ I \end{bmatrix}$$

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{n \times r}$$

The error transmitted code Y is a 1X6 size vector.

$$Y = [1 0 1 0 1 1]$$

$$S = Y \cdot H^T = [1\ 0\ 1\ 0\ 1\ 1] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$(S)_q = (0 0 1)_3$ (syndrome vector)

SL NO.	Error vector E (with single bit error pattern).	syndrome vectors.	Relation with H^T.
1.	0 0 0 0 0 0	0 0 0	- - - -
2.	1 0 0 0 0 0	1 1 1	1st Row of H^T
3	0 1 0 0 0 0	1 1 0	2nd Row of H^T
4.	0 0 1 0 0 0	1 0 1	3rd Row of H^T.
5	0 0 0 1 0 0	1 0 0	4th Row of H^T
6.	0 0 0 0 1 0	0 1 0	5th Row of H^T.
7.	0 0 0 0 0 1	0 0 1	6th Row of H^T

v) To obtain the decoding of the received words.

$$Y_1 = [101100]$$

$$S = Y_1 \cdot H^T \Rightarrow [101100] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = [110]$

then Error vector is $E = [0\ \textcircled{1}\ 0\ 0\ 0\ 0]$

$X_1 = Y_1 \oplus E = [101100] \oplus [010000]$

$X_1 = [111100]$

then $Y_2 = 000110$

$$S = Y_2 \cdot H^T \Rightarrow [000110] \begin{bmatrix} 111 \\ 110 \\ 101 \\ 100 \\ 010 \\ 001 \end{bmatrix}$$

$S = [110]$

then $E = [0\ \textcircled{1}\ 0000]$

$X_2 = Y_2 \oplus E = [000110] \oplus [010000]$

$X_2 = [010110]$

X_1 & X_2 are the correct transmitted word

→ If the (7,4) hamming code, the parity check matrix

$$H = \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

i) construct the G.

ii) The code word that begins with 1010

iii) If the received word Y is 0111100, then the decode this received codeword.

solutions :- i) To obtain Generator matrix.

i) $$H = \left[\begin{array}{c|c} 1011 & 100 \\ 1101 & 010 \\ 0111 & 001 \end{array}\right] \Rightarrow H = [P^T \mid I]$$

then $$P^T = \begin{bmatrix} 1011 \\ 1101 \\ 0111 \end{bmatrix} \qquad P = \begin{bmatrix} 110 \\ 011 \\ 101 \\ 111 \end{bmatrix}$$

$$G = \left[I_{K \times K} \,;\, P_{K \times q} \right]_{1 \times n}.$$

$$G = \left[I_{4 \times 4} \,;\, P_{4 \times 3} \right]_{1 \times 7}$$

$$G = \left[\begin{array}{c|c} 1000 & 110 \\ 0100 & 011 \\ 0010 & 101 \\ 0001 & 111 \end{array}\right]$$

ii) To obtain the code word begins with 1010.

$m = 1010$

Code word $X = m . G$

$$= 1010 \left[\begin{array}{c|c} 1000 & 110 \\ 0100 & 011 \\ 0010 & 101 \\ 0001 & 1111 \end{array}\right]$$

$$= \left[1010001\right]$$

iii) $Y_1 = 0111100.$

$$H = \left[\begin{array}{c|c} 1011 & 100 \\ 1101 & 010 \\ 0111 & 001 \end{array}\right]$$

Syndrome $S = Y_1 . H^T$

$$S = \left[0111100\right] . \left[\begin{array}{c} 110 \\ 011 \\ 101 \\ 111 \\ 100 \\ 010 \\ 001 \end{array}\right]\begin{array}{l} 1 \\ 2 \\ 3 \Rightarrow \text{error} \\ \\ \\ \\ \\ \end{array} \qquad H^T = \left[\begin{array}{c} 110 \\ 011 \\ 101 \\ 111 \\ 100 \\ 010 \\ 001 \end{array}\right]$$

$= \left[101\right]$ error in 3rd position then error vector

$E = \left[0010000\right]$ then

$X = Y_1 \oplus E \Rightarrow \left[0111100\right] \oplus \left[0010000\right]$

$= 0101100$ (decode received codeword).

ry cyclic codes :-

cyclic codes are the sub class of linear block codes. Cyclic codes can be in systematic or non-systematic form. In systematic form, check bits are calculated separately and the code vector is $X = (M:C)$.

Definition of cyclic code :-

A linear code is called cyclic code if every shift of the code vector produces some other code vector. This definition includes two fundamental properties of cyclic codes.

1. linearity   2. cyclic property..

linearity property :- This property state that sum of any two codewords is also a valid codeword.

$X_3 = X_1 \oplus X_2$ X_1 and X_2 are two codewords.

cyclic property :- cyclic shift of the valid code vector produ another valid code vector. Because of this property, the name 'cyclic' is given. consider an n-bit code vector

$X = \{x_{n-1}, x_{n-2}, --- x_1, x_0\} \rightarrow x_{n-1}$ (MSB), x_0 (LSB)

one cyclic shift of X gives

$X' = \{X_{n-2}, X_{n-3}, ---- X_0, x_{n-1}\}$ X_{n-2} (MSB), X_{n-1} (LSB)

one more cyclic shift of X gives

$X'' = \{x_{n-3}, X_{n-4} ---- X_{n-1}, X_{n-2}\}$ X_{n-3} (MSB), X_{n-2} (LSB.

Algebraic structures of cyclic codes :-

The codewords can be represented by a polynomial.

Consider the n- bit codeword

$$X = (x_{n-1}, x_{n-2}, \dots x_1, x_0).$$

The codeword can be represented by a polynomial of degree less than or equal to $(n-1)$ i.e.

$$X(P) = x_{n-1} P^{n-1} + x_{n-2} P^{n-2} + \dots + x_1 P + x_0$$

P^{n-1} represents MSB, P^1 represents second bit from LSB side

P^0 represents LSB.

why to represent codewords by a Polynomial

i) There are algebraic codes. Hence algebraic operations such as addition, multiplication, division, subtraction etc becor very simple

ii) positions of the bits are represented with the help of powers of P in a polynomial.

Generation of code vectors in Non systematic form :-

Let $M = \{M_{k-1}, M_{k-2}, \dots m_1, m_0\}$ be 'k' bits of message vector

Then it can be represented by the polynomial as.

$$m(P) = m_{k-1} P^{k-1} + m_{k-2} P^{k-2} + \dots + M_1 P + m_0$$

let $X(P)$ represent the code word polynomial. It is given as

$$X(P) = M(P) \cdot G(P).$$

Here $G(P)$ is the generating polynomial of degree q.

For an (n,k) cyclic code, $q = n-k$. represent the number of parity bits.

$$G(P) = P^q + g_{q-1} P^{q-1} + \dots + g_1 P + 1$$

Here $g_{q-1}, g_{q-2} \dots g_1$ are the parity bits.

If $M_1, M_2, M_3 \dots$ etc are the other message vectors, then the corresponding code vectors can be calculated as,

$$X_1(P) = M_1(P) \cdot G(P)$$

$$X_2(P) = M_2(P) \cdot G(P).$$

$$X_3(P) = M_3(P) \cdot G(P).$$

All the above code vectors $X_1, X_2, X_3 \dots$ are in non-systematic form and they satisfy cyclic property.

Generation of code vectors in systematic form :-

The systematic form of the block code is

$X = ($ message bits $:$ $(n-k)$ check bits$)$

$= (m_{k-1}, m_{k-2}, \dots m_1 m_0 : c_{q-1} c_{q-2} \dots c_1 c_0).$

Here the check bits form a polynomial as.

$$C(P) = c_{q-1} P^{q-1} + c_{q-2} P^{q-2} + \dots + c_1 P + c_0.$$

The check bit polynomial is obtained by.

$$C(P) = \text{rem}\left(\frac{P^q M(P)}{G(P)}\right)$$

problem

1. The generator polynomial of a $(7,4)$ cyclic code is $G(P) = P^3 + P$ Find all the code vectors for the code in non systematic form

Solution :- Here $n = 7$ and $k = 4$ therefore $q = n - k = 3.$

$2^k = 2^4 = 16$ message vectors of 7 bits each

consider any message vector as $M = (m_3\ m_2\ m_1\ m_0)$

$= (0101)$

Then the message polynomial will be ($K = 4$)

$$m(P) = m_3 P^3 + m_2 P^2 + m_1 P + m_0$$ 0 1 0 1
3 2 1 0

$$m(P) = P^2 + 1.$$

And given generator polynomial is

$$G(P) = P^3 + P + 1$$

In non-systematic code. Encoded data polynomial is

$$X(P) = m(P).G(P)$$

$$= (P^2+1).(P^3+P+1)$$

$$= P^5 + \cancel{P^3} + P^2 + \cancel{P^3} + P + 1$$

$$= P^5 + P^2 + P + 1$$

that means $= 0P^6 + 1P^5 + 0P^4 + 0P^3 + P^2 + P + 1.$

then $X(P) = [X_6\ X_5\ X_4\ X_3\ X_2\ X_1\ X_0]$

$$= (0\ 1\ 0\ 0\ 1\ 1\ 1)$$

Similary other code vectors.

SL NO	message bits (M)					Non-systematic code vectors:						
	m_3	m_2	m_1	m_0		$X = X_6$	X_5	X_4	X_3	X_2	X_1	X_0
1	0	0	0	0	$\longrightarrow$	0	0	0	0	0	0	0
2.	0	0	0	1	$\longrightarrow$	0	0	0	1	0	1	1
3	0	0	1	0	$\longrightarrow$	0	0	1	0	1	1	0
4.	0	0	1	1	$\longrightarrow$	0	0	1	1	1	0	1
5	0	1	0	0	$\longrightarrow$	0	1	0	1	1	0	0
6	0	1	0	1	$\longrightarrow$	0	1	0	0	1	1	1
7	0	1	1	0	$\longrightarrow$	0	1	1	1	0	1	0
8	0	1	1	1	$\longrightarrow$	0	1	1	0	0	0	1
9	1	0	0	0	$\longrightarrow$	1	0	1	1	0	0	0
10	1	0	0	1	$\longrightarrow$	1	0	1	0	0	1	1
11	1	0	1	0	$\longrightarrow$	1	0	0	1	1	1	0
12	1	0	1	1	$\longrightarrow$	1	0	0	0	1	0	1
13	1	1	0	0	$\longrightarrow$	1	1	1	0	1	0	0
14	1	1	0	1	$\longrightarrow$	1	1	1	1	1	1	1

1 1 1 0	1 1 0 0 0 1 0
1 1 1 1	1 1 0 1 0 0 1

Generator & parity check matrices of cyclic codes :-

1. Non - Systematic form of generator matrix :-

Let the $G(P)$ be the generator matrix of size $K \times n$.

So $G(P) = P^{q} + g_{P-1} P^{q-1} + g_{P-2} P^{q-2} + \dots g_1 P + 1$

multiply on both sides of this polynomial by P^i

$$P^i G(P) = P^{i+q} + g_{q-1} P^{i+q-1} + g_{q-2} P^{i+q-2} \dots + g_1 P^{i+1} + P^i$$

$i = (K-1), (K-2), \dots 2, 1, 0.$

Each row of generator matrix can be obtained by changing the values of i.

Systematic form of generator matrix :-

The systematic form of generator matrix is given as

$$G = (I_K ; P_{K \times q})_{K \times n}.$$

The t^{th} row of generator matrix can be given as

$$\boxed{P^{n-t} + R_t(P) = Q_t(P) . G(P)}$$

→ The generator polynomial of a (7,4) cyclic code is $G(P) = P^3 + P + 1$. Find all the code vectors for the code in systematic form.

Solution :-

Here $n = 7$ and $K = 4$, $q = n - K = 3$.

$2^K = 2^4 = 16$ message vectors of 7 bits each.

$M = (m_3\, m_2\, m_1\, m_0)$

$= (1\ 0\ 1\ 0)$

$m(P) = m_3 P^3 + m_2 P^2 + m_1 P + m_0$

$= P^3 + P.$

And given generator polynomial is, $G(P) = P^3+P+1$

To obtain check bits $C(P)$:-

$$C(P) = \text{rem}\left[\frac{P^q\, m(P)}{G(P)}\right]$$

$q = 3$

$m(P) = P^3+P$

$G(P) = P^3+P+1$

$$C(P) = \text{rem}\left(\frac{P^3.(P^3+P)}{P^3+P+1}\right) \Rightarrow$$

$P^3+P+1)\ \cancel{P^6}+\cancel{P^4}\ (P^3+1$

$\cancel{P^6}+\cancel{P^4}+P^3$

$\cancel{P^3}$

$\cancel{P^3}+P+1$

$P+1$

$$= \text{rem}\left(\frac{P^6+P^4}{P^3+P+1}\right)$$

$C(P) = P+1$

$C = 011$

SL NO	message bits.				Systematic code vectors.						
	M = m_3	m_2	m_1	m_0	m_3	m_2	m_1	m_0	C_2	C_1	C_0.
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	1	0	1	1
3	0	0	1	0	0	0	1	0	1	1	0
4	0	0	1	1	0	0	1	1	1	0	1
5	0	1	0	0	0	1	0	0	1	1	1

6	0 1 0 1	→	0 1 0 1 1 0 0
7	0 1 1 0	→	0 1 1 0 0 0 1
8	0 1 1 1	→	0 1 1 1 0 1 0
9	1 0 0 0	→	1 0 0 0 1 0 1
10.	1 0 0 1	→	1 0 0 1 1 1 0
11.	1 0 1 0	→	1 0 1 0 0 1 1
12.	1 0 1 1	→	1 0 1 1 0 0 0
13.	1 1 0 0	→	1 1 0 0 0 1 0
14.	1 1 0 1	→	1 1 0 1 0 0 1
15.	1 1 1 0	→	1 1 1 0 1 0 0
16.	1 1 1 1	→	1 1 1 1 1 1 1.

→ Find out the generator matrix for a systematic (7,4) cyclic code if $G(P) = P^3 + P + 1$. Also find out the parity check matrix.

Solution :- To obtain the Generator matrix is given by

$$P^{n-t} + R_t(P) = Q_t(P) \cdot G(P).$$

In this (7,4) Binary cyclic code then.

$n = 7$, $K = 4$ and $q = 7 - 4 = 3$.

→ if $t = 1$, $G(P) = P^3 + P + 1$

$$P^{7-1} + R_t(P) = Q_t(P) \cdot P^3 + P + 1 \Rightarrow P^6 + R_t(P) = Q_t(P) \cdot P^3 + P + 1$$

$$P^3+P+1) \cancel{P^6} \; (P^3 + P + 1$$

$$\underline{\cancel{P^6} + P^4 + P^3}$$

$$\cancel{P^4} + P^3$$

$$\underline{\cancel{P^4} + P^2 + P}$$

$$\cancel{P^3} + P^2 + \cancel{P}$$

$$\underline{\cancel{P^3} + \cancel{P} + 1}$$

$$P^2 + 1$$

$R_t(P) = P^2 + 1$

$Q_t(P) = P^3 + P + 1$

$p^6+p^2+1 = (p^3+p+1)(p^3+p+1)$

$p^6+p^2+1 = p^6+\cancel{p^4}+\cancel{p^3}+\cancel{p^4}+p^2+\cancel{p}+\cancel{p^3}+\cancel{p}+1$

$\boxed{p^6+p^2+1 = p^6+p^2+1}$ 1st Row polynomials.

$$\begin{array}{l} p^3+p+1)\ \cancel{p^5}\ (p^2+1 \\ \quad \cancel{p^5}+p^3+p^2 \\ \hline \quad \cancel{p^3}+p^2 \\ \quad \cancel{p^3}+p+1 \\ \hline \end{array}$$

$R_t(P) = p^2+p+1$ $\quad p^2+P+1$

$Q_t(P) = p^2+1$

if $t=2$,

$p^{7-2}+R_t(P) = Q_t(P).G(P)$

$p^5+R_t(P) = Q_t(P).\ p^3+P+1$

$p^5+p^2+P+1 = (p^2+1)(p^3+P+1)$

$p^5+p^2+P+1 = p^5+\cancel{p^3}+p^2+\cancel{p^3}+P+1$

$\boxed{p^5+p^2+P+1 = p^5+p^2+P+1}$ 2nd row polynomials

→ if $t=3$.

$$\begin{array}{l} p^3+p+1)\ \cancel{p^4}\ (P \\ \quad \cancel{p^4}+p^2+P \\ \hline \quad p^2+P \end{array}$$

$p^{7-3}+R_t(P) = Q_t(P).G(P)$

$p^4+R_t(P) = Q_t(P).(p^3+P+1)$

$p^4+p^2+P = P.(p^3+P+1)$

$\boxed{p^4+p^2+P = p^4+p^2+P}$ → 3rd row polynomials.

→ if $t=4$.

$$\begin{array}{l} p^3+P+1)\ \cancel{p^3}\ (1 \\ \quad \cancel{p^3}+P+1 \\ \hline \quad P+1 \end{array}$$

$p^{7-4}+R_t(P) = Q_t(P).G(P)$

$p^3+R_t(P) = Q_t(P).(p^3+P+1)$

$\boxed{p^3+P+1 = 1.(p^3+P+1)}$ → 4th row polynomials.

Then Generator matrix is

$$G = \begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \end{array} \begin{array}{c} \begin{array}{ccccccc} p^6 & p^5 & p^4 & p^3 & p^2 & p^1 & p^0 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]_{4\times 7} \end{array}$$

ded data $X = m.G$

$m = m_3 m_2 m_1 m_0$

$= 1\ 1\ 0\ 0$

$$X = m.G = \begin{bmatrix}1100\end{bmatrix} \begin{bmatrix} 1000 & 101 \\ 0100 & 111 \\ 0010 & 110 \\ 0001 & 011 \end{bmatrix}$$

$$= \begin{bmatrix}1100010\end{bmatrix}$$

To obtain parity check matrix (H)

The parity check matrix H is given by

$$H = \begin{bmatrix} P^T \vdots I_q \end{bmatrix}_{q \times n}$$

The parity matrix in generator matrix is $P = \begin{bmatrix} 101 \\ 111 \\ 110 \\ 011 \end{bmatrix}$

then $P^T = \begin{bmatrix} 1110 \\ 0111 \\ 1101 \end{bmatrix}$

I_q is the $q \times q$ Identity matrix. $I_q = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$

Then parity check matrix is $H = \begin{bmatrix} 1110 & 100 \\ 0111 & 010 \\ 1101 & 001 \end{bmatrix}$

This is the required parity check matrix for (7,4) cyclic code in systematic form.

→ A message 101101 is to be transmitted in cyclic code w generator polynomial $G(d) = d^4+d^3+1$. obtain the transm code word. How many check bits does the encoded mess contain.

Solution :- Given $M = \underset{5}{1}\underset{4}{0}\underset{3}{1}\underset{2}{1}\underset{1}{0}\underset{0}{1}$ $K = 6$.

Then message polynomial.

$$m(P) = m_5 P^5 + m_4 P^4 + m_3 P^3 + m_2 P^2 + m_1 P^1 + m_0 P^0$$

$$= 1P^5 + 0P^4 + 1P^3 + 1.P^2 + 0.P^1 + 1.P^0$$

$$m(P). = P^5 + P^3 + P^2 + 1$$

The degree of generator polynomial is four. The degree of generator polynomial is equal to the number of parity bits in the code.

number of parity bits $q = 4$.

Code word length $n = K + q = 10$

$G(d) = d^4+d^3+1$ it can be written as.

$G(P) = P^4 + P^3 + 1$.

To find the check bit polynomial $C(P) = \text{rem}\left(\dfrac{P^q(m(P))}{G(P)}\right)$

$$C(P) = \text{rem}\left(\frac{P^4(P^5+P^3+P^2+1)}{P^4+P^3+1}\right)$$

$$C(P) = \text{rem}\left(\frac{P^9+P^7+P^6+P^4}{P^4+P^3+1}\right)$$

$$p^4+p^3+1 \overline{\big)\, \cancel{p^9}+p^7+p^6+p^4} \big(p^5+p^4+p^2$$

$$\cancel{p^9}+p^8+p^5$$

$$\cancel{p^8}+\cancel{p^7}+p^6+p^5+\cancel{p^4}$$

$$\cancel{p^8}+\cancel{p^7}+\cancel{p^4}$$

$$\cancel{p^6}+\cancel{p^5}$$

$$\cancel{p^6}+\cancel{p^5}+p^2$$

$$p^2$$

remainder $= p^2$

$C(P) = p^2$ $\quad \underset{3}{0}\underset{2}{1}\underset{1}{0}\underset{0}{0}$

$C = 0010$

The Transmitted code word $X = [m_{K-1} \ldots M0, c_{q-1} \ldots c_0]$

Then codeword $X = [m_5, m_4, m_3, m_2, m_1, m_0 : c_3\, c_2\, c_1\, c_0]$

$$X = [\underset{9}{1}\underset{8}{0}\underset{7}{1}\underset{6}{1}\underset{5}{0}\underset{4}{1} : \underset{3}{0}\underset{2}{1}\underset{1}{0}\underset{0}{0}]$$

The codeword polynomial is

$$X(P) = X_9 P^9 + X_8 P^8 + X_7 P^7 + X_6 P^6 + X_5 P^5 + X_4 P^4 + X_3 P^3 + X_2 P^2 + X_1 P^1 + X_0 P^0$$

$$= 1 \times P^9 + 0 \times P^8 + 1 \times P^7 + 1 \times P^6 + 0 \times P^5 + 1 \times P^4 + 0 \times P^3 + 1 \times P^2 + 0 \times P^1 + 0 \times P$$

$$= P^9 + P^7 + P^6 + P^4 + P^2$$

→ For a generator polynomial P^3+P+1 of a (7,4) cyclic code. Determine the data vectors transmitted for the following received vectors.

i) 1101101

ii) 0101000.

Sol $G = \left[\begin{array}{cccc|ccc} 1&0&0&0&1&0&1 \\ 0&1&0&0&1&1&1 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&0&1&1 \end{array}\right]$

$H^T = \begin{bmatrix} 101 \\ 111 \\ 110 \\ 011 \\ \boxed{100} \\ 010 \\ 001 \end{bmatrix}$

Solution :-

i) $Y = \underset{6543210}{1101101}$

$Y(P) = P^6+P^5+P^3+P^2+1$

then

$S = \text{rem}\left(\frac{Y(P)}{G(P)}\right)$

$S(P) = \text{rem}\left[\frac{P^6+P^5+P^3+P^2+1}{P^3+P+1}\right]$

$$P^3+P+1)\ \cancel{P^6}+P^5+\cancel{P^3}+P^2+1\ (P^3+P^2+P+1$$

$$\cancel{P^6}+P^4+\cancel{P^3}$$

$$\cancel{P^5}+P^4+\cancel{P^2}+1$$

$$\cancel{P^5}+P^3+\cancel{P^2}$$

$$\cancel{P^4}+P^3+1$$

$$\cancel{P^4}+P^2+P$$

$$\cancel{P^3}+P^2+\cancel{P}+\cancel{1}$$

$$\cancel{P^3}+\cancel{P}+\cancel{1}$$

$$P^2$$

$S(P) = P^2 \Rightarrow 100$

Error in 5th position.

$X = Y \oplus E = [1101101] \oplus [0000100]$

$= [1101001]$

ii) $Y = \underset{6543210}{0101000}$

$Y(P) = P^5+P^3$.

$S = \text{rem}\left[\frac{P^5+P^3}{P^3+P+1}\right]$

$$P^3+P+1)\ \cancel{P^5}+\cancel{P^3}\ (P^2$$

$$\cancel{P^5}+\cancel{P^3}+P^2$$

$$P^2$$

$S(P) = P^2$

$S = 100$ Error in 5th position.

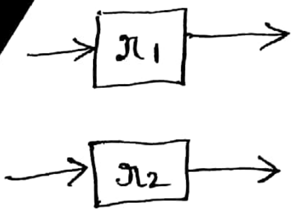
$X = Y \oplus E$

$= [0101000] \oplus [0000100]$

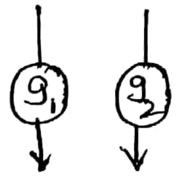
$= [0101100]$.

der using an (n-K) Bit Shift Register :-

The symbols used to draw encoders are.

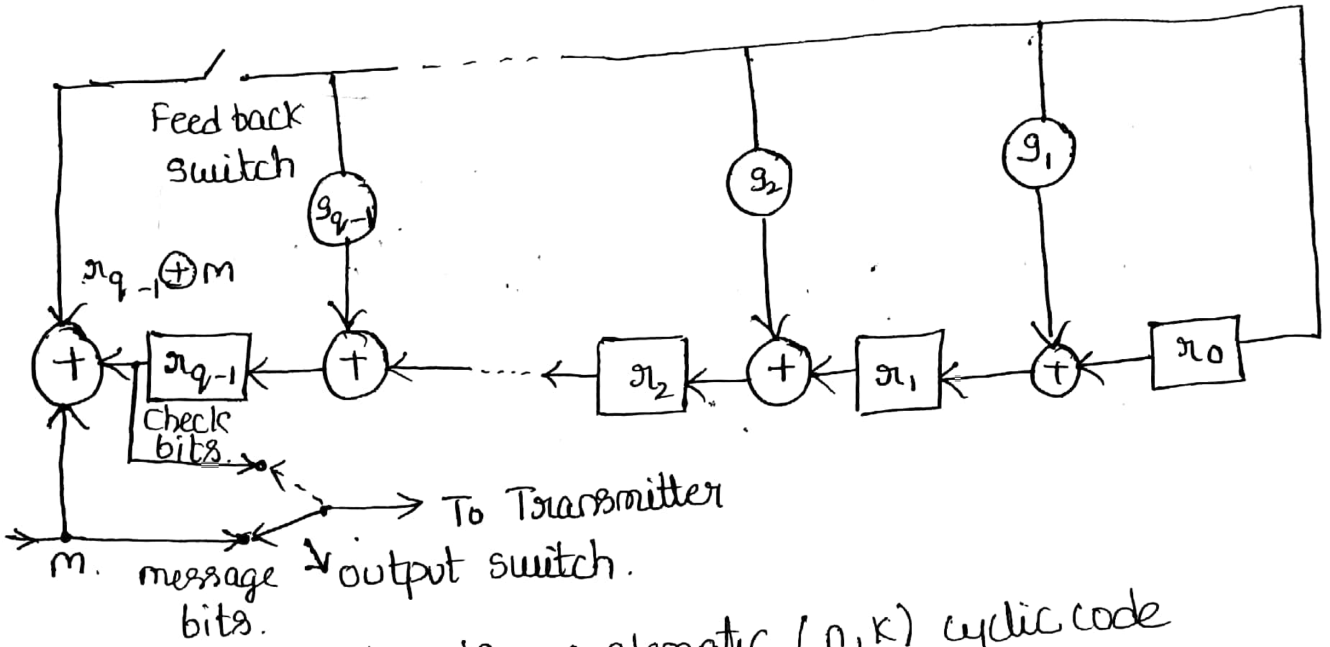


These are flip-flops. They are connected in sequential order to make a shift register. The contents of the shift register are shifted from input to output when clock pulse is applied.



They represent closed path if g = 1 and open path if g = 0.

These symbols represent mod-2 addition.



Encoder for systematic (n, K) cyclic code

operation :- The feedback switch is first closed. The K message bits are shifted to the transmitter as well as shifted into the registers. The output switch is connected to message input. All the shift register are initialized to all zero state.

After the shift of 'K' message bits the registers contain 'q' check bits. The feedback switch is now opened and output switch is connected to check bits position.

with the every shift, the check bits are then shifte the transmitter. The Block diagram performs the divi operation and generates the remainder. The remainder stored in the shift register after all message bits are shifted out.

→ Design the encoder for the (7,4) cyclic code generated by $G(P) = P^3 + P + 1$ and verify its operation for any message vector.

solution:- The generator polynomial is

$$G(P) = P^3 + P^2 \cdot 0 + P + 1$$

$$G(P) = P^3 + g_2 P^2 + g_1 P + 1$$

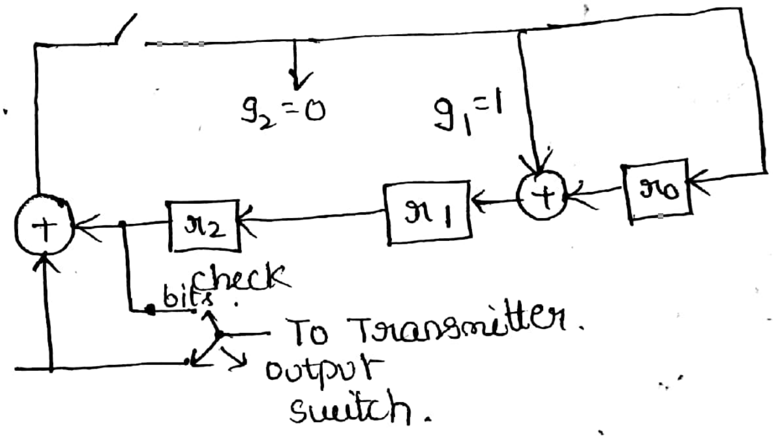
$g_1 = 1$ and $g_2 = 0$

$q = n - K = 7 - 4 = 3.$

consider any message vector

$M = M_3 m_2 m_1 m_0$

$= 1100.$



Since $q = 3$ there are '3' flip flops in shift register to hold check bits C_1 C_2 and C_0. Since $g_2 = 0$, its link is not connected. $g_1 = 1$, hence its link is connected.

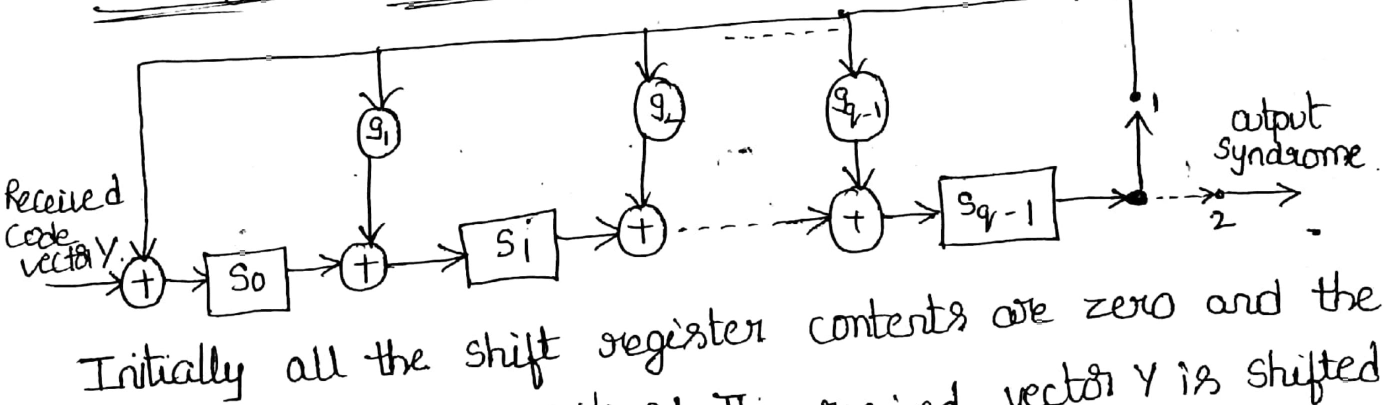
Input message bit (m).	Register bit inputs before shift			Register bit outputs after shift		
	$r_2 = r_2'$	$r_1 = r_1'$	$r_0 = r_0'$	$r_2' = r_1$	$r_1' = r_0 \oplus r_2 \oplus m$	$r_0' = r_2 \oplus M$
	0	0	0	0	0	0
1	0	0	0	0	$0 \oplus 0 \oplus 1 = 1$	$0 \oplus 1 = 1$
	0	1	1	1	$1 \oplus 0 \oplus 1 = 0$	$0 \oplus 1 = 1$
0	1	0	1	0	$1 \oplus 1 \oplus 0 = 0$	$1 \oplus 0 = 1$
0	0	0	1	0	$1 \oplus 0 \oplus 0 = 1.$	$0 \oplus 0 = 0$

Table shows the contents of shift registers before and fter shifts. The table shows that at the end of last messag bit the register bit outputs are $r_2=0$, $r_1'=1$ and $r_0'=0$. The feed back switch is opened and output switch is closed to check bits position. The check bits are shifted as $C_2=r_2'$, $C_1=r$ and $C_0=r_0'$. then $X=(m_3 m_2 m_1 m_0 C_2 C_1 C_0) = (1100010)$

operation of (7,4) cyclic code Encoder :-

Shift clock	message bit m	shift register o/p			Feed back Switch ON/OFF	o/p switch position	Transmitted bits.
		r_2'	r_1'	r_0'			
1	1	0	1	1	ON	message	1
2	1	1	0	1	on	message	1
3	0	0	0	1	on	message	0
4	0	0	1	0	on	message	0
5	-	0	1	0	off	checkbits	0
6	-	1	0	0	off	checkbits	1
7	-	0	0	0	off	checkbits	0

Block diagram of Syndrome calculator :-



Initially all the shift register contents are zero and the switch is closed in position 1. The received vector Y is shifted bit by bit into the shift register. The contents of flip-flops keep on changing according to input bits of Y and values of g_1, g_2 etc. After all the bits of Y are shifted, the 'q' flip-flops of shift register contains the q-bit syndrome vector.

The switch is then closed to position 2 and cloc applied to the shift register. The output is a syndrome vector $S = (S_{q-1}, S_{q-2}, \dots S_1, S_0)$.

1. Design a syndrome calculator for a (7,4) cyclic Hamming code generated by the polynomial $G(P) = P^3+P+1$. Calculate the syndrome for $Y = (1001101)$.

Solution:- (7,4) cyclic Hamming code

$n=7, k=4, q = 7-4$
$= 3.$

The generator polynomial is $G(P) = P^3 + P^{\cancel{2}} + 1$

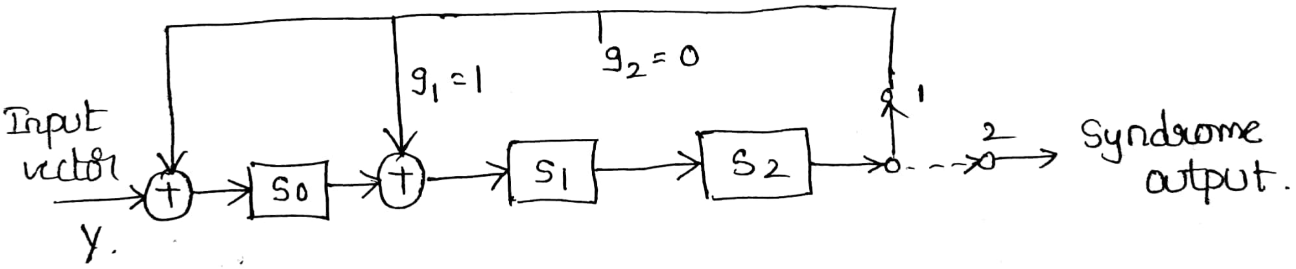
it can be written as. $G(P) = P^3 + 0P^2 + P + 1$.

$G(P) = P^3 + g_2P^2 + g_1P + 1$

on comparison of the above two equations.

$g_1 = 1, \quad g_2 = 0.$

Block diagram :-



Shift	Received vector (Y).	contents of flipflops in shift register		
-	-	$S_0 = Y \oplus S_2$	$S_1 = S_0 \oplus S_2$	$S_2 = S_1$
-	-	0	0	0
1	1	$1 \oplus 0 = 1$	0	0
2	0	0	$1 \oplus 0 = 1$	0
3	0	0	0	1
4	1	0	1	0
5	1	1	0	1
6	0	1	0	0
7	1 (syndrome)	1	1	0

Syndrome vector $S = S_2 S_1 S_0 = 011$

The generator polynomial of a (15, 11) hamming code is given by $g(x) = 1 + x + x^4$. Develop encoder & syndrome calculator for this code using systematic form.

Solution :-

(15, 11) hamming code.

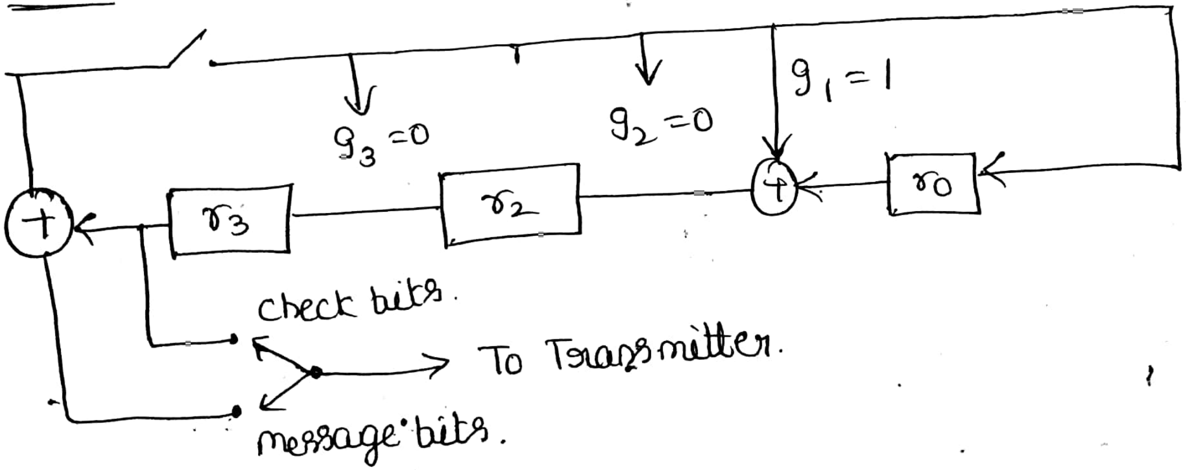
$n = 15, \ K = 11, \ q = 4.$

$g(x) = x^4 + x + 1$

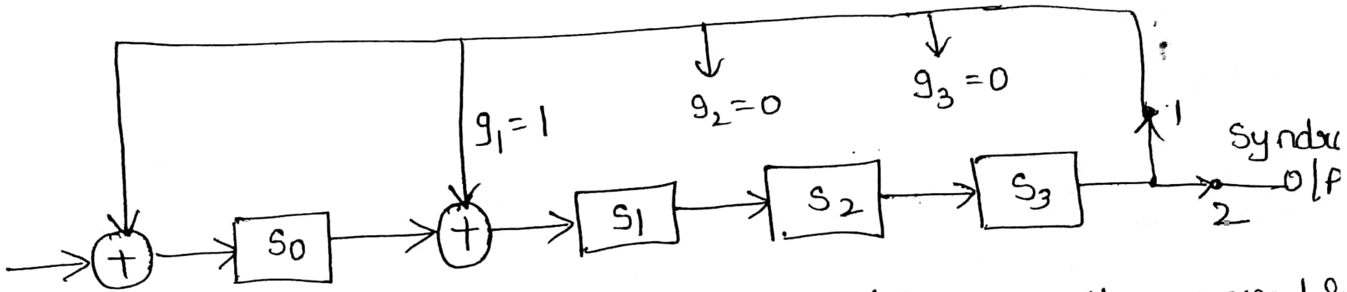
it can be written as $g(x) = x^4 + g_3 x^3 + g_2 x^2 + g_1 x + 1$..

$g_3 = 0, \ g_2 = 0, \ g_1 = 1$

Encoder :-



Syndrome calculator :-



3. sketch the encoder & syndrome calculator for the generato polynomial $g(x) = 1 + x^2 + x^3$. & obtain the syndrome for the received code word 1001011.

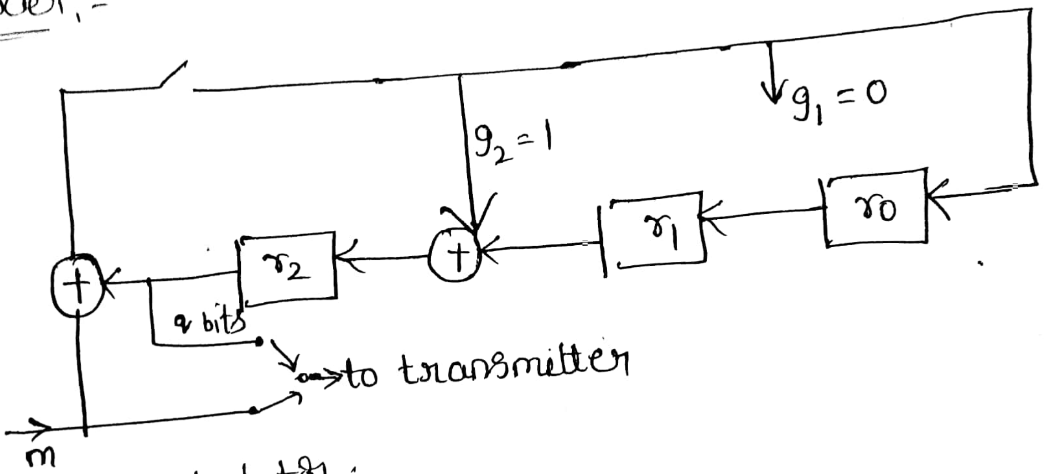
Solution :-

$g(x) = 1+x^2+x^3$

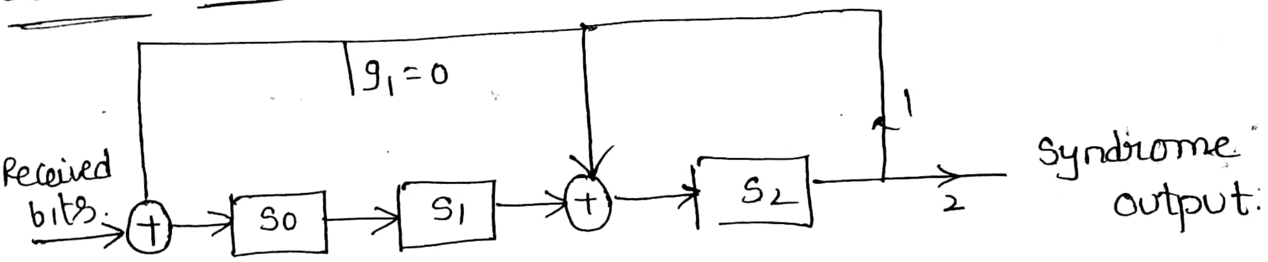
$q = 3, \quad g(x) = x^3 + g_2 x^2 + g_1 x^1 + 1\,x^0$

$g_2 = 1, \quad g_1 = 0$

Encoder :-



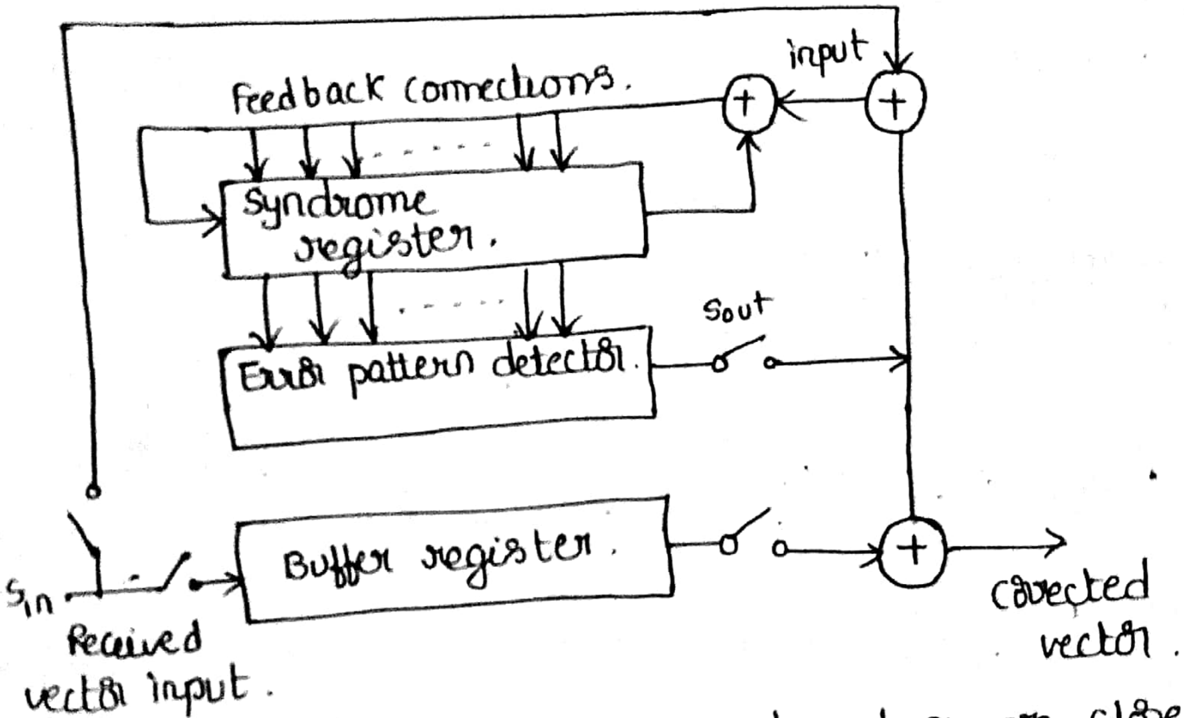
Syndrome calculator :-



Shift	Received vector i.e bits of Y	Contents of flipflop in shift regis		
		$S_0 = y \oplus S_2$	$S_1 = S_0$	$S_2 = y \oplus S_1$
-	-	0	0	0
1	1	1	0	1
2	0	1	1	0
3	0	0	1	1
4	1	0	0	0
5	0	0	0	0
6	1	1	0	1
7	1	0	1	1

...ter for cyclic codes :-

...nce the syndrome is calculated, then an error pattern is detected for that particular syndrome. when this error vector is added to the received vector Y, then it gives corrected code vector at the output.



→ The switches named Sout are opened and Sin are closed. The bits of the received vector Y are shifted into the buffer register as well as they are shifted into the syndrome calculator.

→ when all the 'n' bits of the received vector Y are shifted into buffer register and syndrome calculator the syndrome register holds a syndrome vector.

→ The syndrome vector is given to the error pattern detector. A particular syndrome detects a specific error pattern. The switches Sin are opened and Sout are closed.

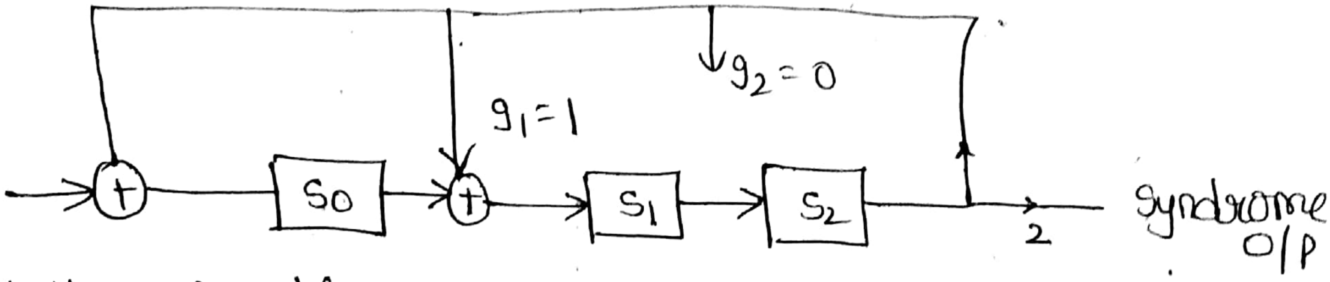
→ The error pattern is then added bit by bit to the received vector. The output is the corrected error free vector.

4. A binary message sequence 1001 is coded using a g
polynomial $G(X) = X^3+X+1$. Assuming a systematic cycl
coding is used, determine the transmitted code word. & If
4^{th} bit in the received word is in error, workout the
Syndrome & draw the hardware block diagram for the
syndrom generator. (Tx) = 1001 110)

$$G(X) = X^3+X+1, \quad G(X) = X^3+g_2x^2+g_1x^1+1$$

$$q = 3. \qquad g_2 = 0, \quad g_1 = 1$$

Syndrome calculator :-



Shift clock.	Rx vector Y.	$S_0 = Y \oplus S_2$	$S_1 = S_0 \oplus S_2$	$S_2 = S_1$
-	-	0	0	0
	1	1	0	0
	0	0	1	0
	0	0	0	1
	0	1	1	0
	1	1	1	1
	1	0	0	1
	0	1	1	0

$S = S_2 S_1 S_0 = 011.$

4^{th} bit error Rx code word = (1000 110).

error in 4^{th} position.

H codes :- (Bose-chaudhri-Hocquenghem codes).

-BCH codes are most extensive and powerful error correcting cyclic codes.

2. For any positive integer m and t (where $t < (2^m-1)/2$) there exists a BCH code.

Block length : $n = 2^m-1$

number of parity check bits : $n-k \leq mt$.

minimum distance $d_{min} \geq 2t+1$

3. Each BCH code can detect and correct upto 't' random errors.
4. BCH codes have flexibility in selection of block length and code rate.
5. BCH codes are the best code for block lengths upto few hundred bits.
6. The decoding schemes of BCH codes can be implemented on digital computer.
7. Because of software implementation of decoding schemes they are quite flexiable compared to hardware implementat of other schemes.

non - systematic form :-

$G(P) = P^3+P+1 \Rightarrow P^iG(P) = P^{i+3}+P^{i+1}+P^i$

obtain the generator matrix corresponding to $G(P) = P^3+P^2+1$ for a (7,4) cyclic code.(non-systematic form)

solution :-

Here, $n=7$, $k=4$ and $q = 7-4 = 3$.

$P^iG(P) = P^{i+3}+P^{i+2}+P^i$

$k-1 = 3$, $i = 3,2,1,0$.

For row 1 : i = 3 ⇒ $P^3 . G(P) = P^6 + P^5 + P^3$

For row 2 : i = 2 ⇒ $P^2 . G(P) = P^5 + P^4 + P^2$

For row 3 : i = 1 ⇒ $P . G(P) = P^4 + P^3 + P$

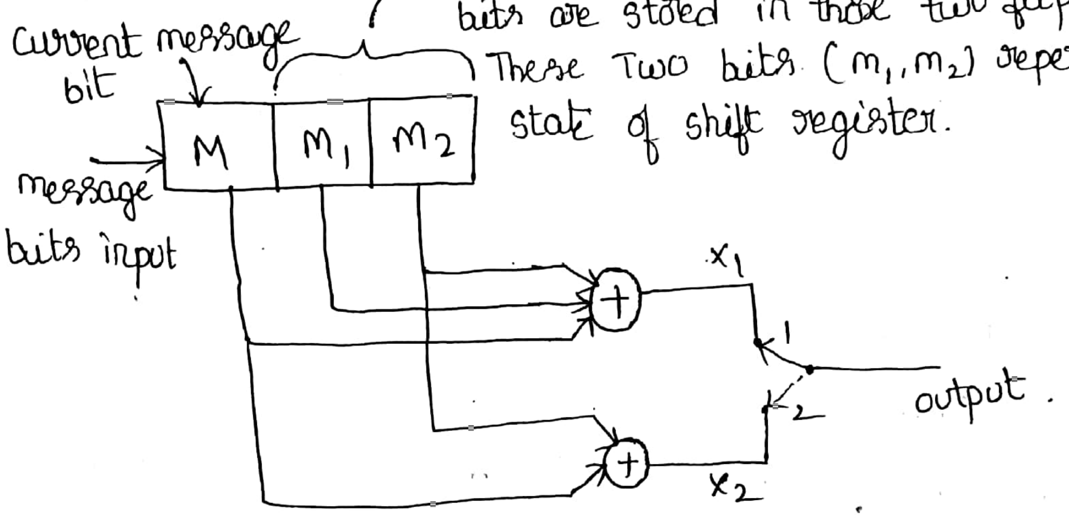
For row 4 : i = 0 ⇒ $G(P) = P^3 + P^2 + 1$

$$\begin{array}{lc|ccccccc|}
 & & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\text{Row } 1 \rightarrow P^3 G(P) & = & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\text{Row } 2 \rightarrow P^2 G(P) & = & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\text{Row } 3 \rightarrow P . G(P) & = & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\text{Row } 4 \rightarrow G(P) & = & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}$$

4 x 7.

volutional codes :-

A convolutional coding is done by combining the fixed number of input bits. The input bits are stored in the fixed length shift register and they are combined with the help of mod-2 adders. This operation is equivalent to binary convolution and hence it is called convolutional coding.



previous two successive message bits are stored in these two flipflops. These Two bits (m_1, m_2) reperesent state of shift register.

Convolutional Encoder with $K = 3$, and $n = 2$.

operation :-

whenever the message bit is shifted to position 'm', the new values of x_1 and x_2 are generated depending upon m, m_1, and m_2.

$x_1 = m_1 \oplus m_2 \oplus m$ m is current message bit

$x_2 = m \oplus m_2$ m_1, m_2 are the previous message bit

The output switch first samples x_1 and then x_2. The shift register then shifts contents of m_1 and m_2 and content of m to m_1. Next input bit is then taken and stored in m. Again x_1 and x_2 are generated according to this new combination of m, m_1 and m_2.

The output switch then samples x_1 then x_2. Then outp

$$X = x_1 x_2 \, x_1 \, x_2 \, x_1 \, x_2 \cdots \text{ and so on}$$

Every input message bit two encoded output bits x_1 and
are transmitted. for a single message bit, the encoded code word is two bits, $K=1$, $n=2$.

Definitions :-

1. Code rate :- The ratio of message bits (K) and the Encod output bits (n) is called code rate

$$r = \frac{k}{n} = \frac{1}{2}$$

whenever a particular message bit enters a shift register it remains in the shift register for three shifts.

→ First shift → message bits is entered in position 'm'

→ Second shift → message bits is shifted in position m_1.

→ Third shift → message bit is shifted in position m_2.

The fourth shift the message bit is discarded or simply lost by overwriting.

2. Constraint length (K).

It can be defined as the number of shifts over which a single message bit can influence the encoder outpu It is expressed in terms of message bits.

$K = 3$.

3. Dimension of the code :- The dimension of the code is given by n and K. ie 'k' is the number of message bits. n is the encoded output bits. Hence the dimension of the code is (n, K). And such encoder is called (n, K) convolutional Encoder. EX :- (2, 1).

ysis of convolutional Encoders :-

Time domain Approach

2. Transform domain Approach.

→ Time domain Approach :-

let the sequence

$\{ g_0^{(1)}, g_1^{(1)}, g_2^{(1)} \ldots g_m^{(1)} \}$ denote the impulse response of the adder which generates x_1.

similarly $\{ g_0^{(2)}, g_1^{(2)}, g_2^{(2)} \ldots g_m^{(2)} \}$ denote the impulse respons of the adder which generates x_2

These impulse response of the adder which generates a sequence of the code is called generator sequence.

let $M = \{ m_0, m_1, m_2 \ldots \}$, encoder generates the two output sequences x_1 and x_2. These are obtained by convolving the generator sequence with the message sequence. The sequence x_1 is given as $X_1 = X_i^{(1)} = \sum_{L=0}^{M} g_l^{(1)} m_{i-l}$

Here $m_{i-1} = 0$ for all $l > i$ & $i = 0, 1, 2 \ldots$

Similary the sequence x_2 is given as

$$x_2 = x_2^{(i)} = \sum_{L=0}^{M} g_l^{(2)} m_{i-l}$$

→ Transform domain Approach :-

let the impulse responses be represented by polynomials ibl

$g^{(1)}(P) = g_0^{(1)} + g_1^{(1)}(P) + g_2^{(1)} P^2 + \ldots + g_m^{(1)} P^m$ similarly

$g^{2}(P) = g_0^{(2)} + g_1^{(2)}(P) + g_2^{(2)} P^2 + \ldots + g_m^{(2)} P^m$.

The message polynomial is represented by.

$$m(P) = m_0 + m_1 P + m_2 P^2 + \dots + m_{L-1} P^{L-1}$$

The convolution sums are converted to polynomial multiplications in the transform domain.

$$x^{(1)}(P) = g^{(1)}(P) \cdot m(P)$$

$$x^{(2)}(P) = g^{(2)}(P) \cdot m(P).$$

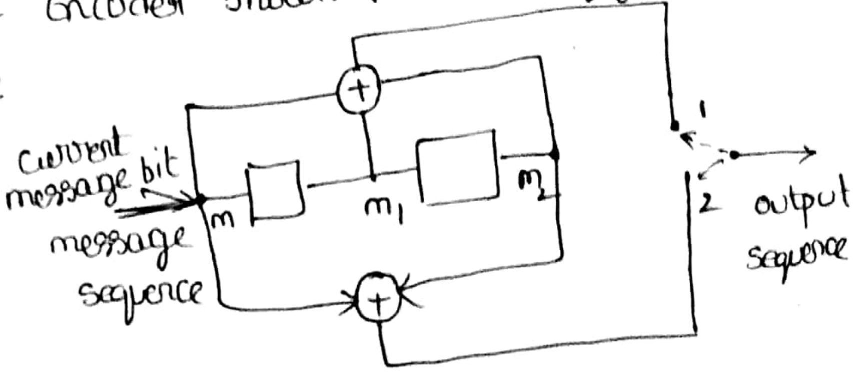
1) For the convolutional Encoder shown in below figure. determ

i) Dimension of the code

ii) code rate

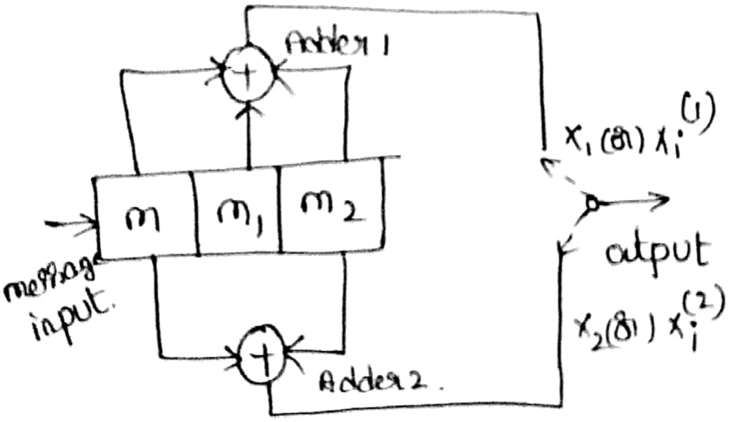
iii) Constraint length

iv) Generating Sequence (impulse response



v) output Sequence for message. sequence of m = {10011}

Solution ;-



i) Dimension of the code.

Dimension = (n, k) = (2, 1)

ii) Code rate

$$r = \frac{k}{n} = \frac{1}{2}$$

iii) constraint length

K = 3bits

iv) Generating sequences

$x_i^{(1)}$ is generated by adding all 3 bits

$$g_i(1) = \{g_0^{(1)}, g_1^{(1)}, g_2^{(1)}\} = \{1, 1, 1\}$$

$x_i^{(2)}$ is generated by adding 1st & last bits $g_i^{(2)} = \{g_0^{(2)}, g_1^{(2)}$

$$= \{1, 0, 1\}$$

$\dot{m} = \{m_0, m_1, m_2, m_3, m_4\} = \{1 0 0 1 1\}$

ime domain approach :-

To obtain output due to adder 1

$$x_i^{(1)} = \sum_{l=0}^{M} g_l^{(1)} m_{i-l}$$

$i=0$ $\quad x_0^{(i)} = \sum_{l=0}^{m} g_l^{(1)} m_{-l} = g_0^{(1)} m_0 = 1.1 = 1.$

$i=1$ $\quad x_1^{(1)} = \sum_{l=0}^{m} g_l^{(1)} m_0 = \overset{g_0^{(1)}.m_1 \oplus g_1^{(1)}.m_0}{1.0 \oplus 1.1} = 1$

$i=2$ $\quad x_2^{(1)} = \sum_{l=0}^{m} g_l^{(1)} m_{2-l}$

$= g_0^{(1)} m_2 \oplus g_1^{(1)} m_1 + g_2^{(1)} m_0$

$= (1\times0) \oplus (1\times0) \oplus (1\times1) = 1$

$i=3.$ $\quad x_3^{(1)} = \sum_{l=0}^{m} g_l^{(1)} m_{3-l}$

$= g_0^{(1)} m_3 \oplus g_1^{(1)} m_2 \oplus g_2^{(1)} m_1$

$= (1\times1) \oplus (1\times0) \oplus (1\times0)$

$= 1.$

$i=4.$ $\quad x_4^{(1)} = \sum_{l=0}^{m} g_l^{(1)} g_{4-l} = g_0^{(1)} m_4 \oplus g_1^{(1)} m_3 \oplus g_2^{(1)} m_2$

$= 1\times1 \oplus 1\times1 \oplus 1\times0 = 0.$

$i=5$ $\quad x_5^{(1)} = g_0^{(1)} m_5 + g_1^{(1)} m_4 \oplus g_2^{(1)} m_3 \oplus = g_1^{(1)} m_4 \oplus g_2^{(1)} m_3$

$= (1\times1) \oplus (1\times1)$

m_5 is not avaliabl

$= 0.$

$i=6$ $\quad x_6^{(1)} = g_0^{(1)} m_6 \oplus g_1^{(1)} m_5 \oplus g_2^{(1)} m_4$

$= g_2^{(1)} m_4 \Rightarrow 1\times1$

$\Rightarrow 1$

m_6 and m_5 is not avaliable

The output of adder 1 is

$x_1 = x_i^{(1)} = \{1111001\}$

To obtain output due to adder 2:-

$$x_2 = x_i^{(2)} = \sum_{l=0}^{m} g_l^{(2)} m_{i-l} \qquad m_{i-l} = 0 \text{ for all } l > i$$

i=0

$x_0^{(2)} = g_0^{(2)} m_0 : 1 \times 1 = 1.$

i=1

$x_1^{(2)} = g_0^{(2)} m_1 + g_0^{(1)} m_0 = (1 \times 0) \oplus 0 \times 1 = 0$

i=2 $x_2^{2} = g_0^{(2)} m_2 + g_1^{(2)} m_1 \oplus g_2^{(2)} m_0 = 1 \times 0 \oplus 0 \times 0 \oplus 1 \times 1$

$= 1$

i=3. $x_3^{(2)} = g_0^{(2)} m_3 + g_1^{(2)} m_2 \oplus g_2^{(2)} m_1 = 1 \times 1 \oplus 0 \times 0 \oplus 1 \times 0$

$= 1$

i=4 $X_4^{(2)} = g_0^{(2)} m_4 \oplus g_1^{(2)} m_3 \oplus g_2^{(2)} m_2 = 1 \times 1 \oplus 0 \times 1 \oplus 1 \times 0$

$= 1$

i=5 $X_5^{(2)} = g_0^{(2)} m_5 \oplus g_1^{(2)} m_4 \oplus g_2^{(2)} m_3$

$= (0 \times 1) \oplus (1 \times 1) = 1.$ (m_5 is not avaliat

i=6 $x_6^{(2)} = g_0^{(2)} m_6 \oplus g_1^{(2)} m_5 \oplus g_0^{(2)} m_4$ (m_5, m_6 are not avaliable).

$= 1 \times 1 = 1$

Then the sequence is

$x_2 = x_i^{(2)} = \{1011111\}$

To obtain multiplexed sequence of x_1 and x_2

$x_i = x_0^{(1)} x_0^{(2)} x_1^{(1)} x_1^{(2)} x_2^{(1)} x_2^{(2)} x_3^{(1)} x_3^{(2)} x_4^{(1)} x_4^{(2)} x_5^{(1)} x_5^{(2)} x_6^{(1)} x_6^{(2)}$

$= \{11, 10, 11, 11, 01, 01, 11\}$

nsform domain Approach :-

The generating polynomial for adder (1) $g_i^{(1)} = \{\underset{0}{1}, \underset{1}{1}, \underset{2}{1}\}$

$g^1(P) = 1+P+P^2$

generating polynomial for adder (2) $g_i^{(2)} = \{\underset{0}{1}, \underset{1}{0}, \underset{2}{1}\}$

$g^2(P) = 1+P^2$

message vector $= \{\underset{0}{1}\underset{1}{0}\underset{2}{0}\underset{3}{1}\underset{4}{1}\}$

message polynomial $= 1+P^3+P^4$

To optain the output due to the adder 1

$$x^1(P) = m(P) \cdot g^1(P)$$

$$= (1+P^3+P^4)(1+P+P^2)$$

$$= 1+P+P^2+P^3+\cancel{P^4}+\cancel{P^5}+\cancel{P^4}+\cancel{P^5}+P^6$$

$$= 1+P+P^2+P^3+P^6$$

$$x_i^{(1)} = \{\underset{0}{1}\underset{1}{1}\underset{2}{1}\underset{3}{1}\underset{4}{0}\underset{5}{0}\underset{6}{1}\}$$

To obtain the output due to the adder 2

$$x^2(P) = m(P) \cdot g^{(2)}(P)$$

$$= (1+P^3+P^4)(1+P^2)$$

$$= 1+P^3+P^4+P^2+P^5+P^6$$

$$= 1+P^2+P^3+P^4+P^5+P^6$$

$$x_i^{(2)} = \{1011111\}$$

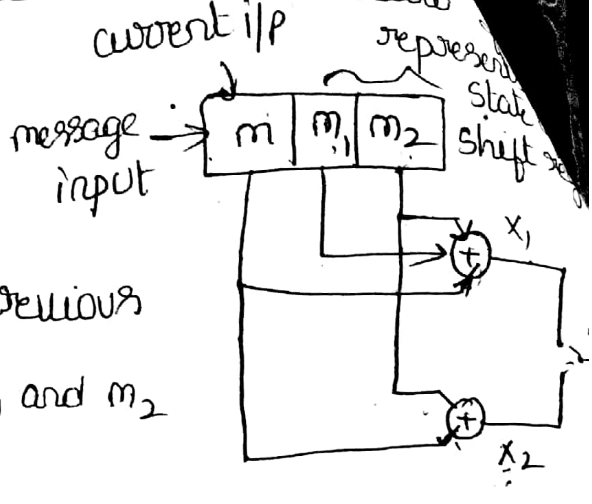
multiplexed output sequence

$$\{x_i\} = \{11, 10, 11, 11, 01, 01, 11\}$$

Code Tree, Trellis and state diagram for a Convolution

Consider the convolutional Encoder shown in below

The input message bit 'm' affects state of encoder as well as outputs x_1 and x_2. The previous two successive message bits m_1 and m_2 represents state.



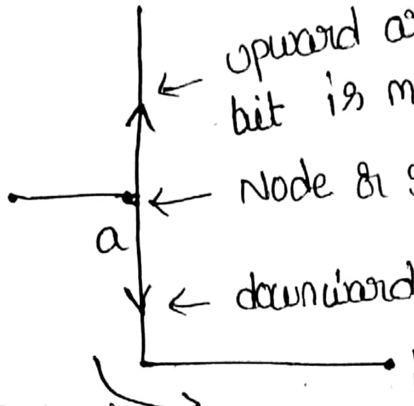
m_2	m_1	State of Encoder
0	0	a.
0	1	b.
1	0	c.
1	1	d.

Fig - State of the Encoder.

Development of the code Tree :-

Intial State	Input message bit	Shift Register. m_0 m_1 m_2	Calculation of x_1 & x_2 $x_1 = m \oplus m_1 \oplus m_2$ $x_2 = m \oplus m_2$	Next state m_2 m_1	outp
a	0	0 | 0 | 0 0 0	$x_1 = 0 \oplus 0 \oplus 0 = 0$ $x_2 = 0 \oplus 0 = 0$	a(0 0)	0 0
	1	1 | 0 | 0 1 0	$x_1 = 1 \oplus 0 \oplus 0 = 1$ $x_2 = 1 \oplus 0 = 1$	b (01)	1
b	0	0 | 1 | 0 0 1	$x_1 = 0 \oplus 1 \oplus 0 = 1$ $x_2 = 0 \oplus 0 = 0$	c.	1.0
	1	1 | 1 | 0	$x_1 = 1 \oplus 1 \oplus 0 = 0$ $x_2 = 1 \oplus 0 = 1$	d	0
c	0	0 | 0 | 1 0 0	$x_1 = 0 \oplus 0 \oplus 1 = 1$ $x_2 = 1 \oplus 0 = 1$	a	1
	1	1 | 0 | 1 1 0	$x_1 = 1 \oplus 0 \oplus 1 = 0$ $x_2 = 1 \oplus 1 = 0$	b.	0
d	0	0 | 1 | 1 0 1	$x_1 = 0 \oplus 1 \oplus 1 = 0$ $x_2 = 0 \oplus 1 = 1$	c	0
	1	1 | 1 | 1	$x_1 = 1 \oplus 1 \oplus = 1$	d	1

intially a be the state of Encoder the upword rrow indicates m=0 & downward arrow indicates m=1



← upward arrow indicates that message bit is m=0

← Node or state

← downward arrow indicates that message bit is m=

b ← This is new state or node when m=1

This indicates output while going from node 'a' to 'b'.

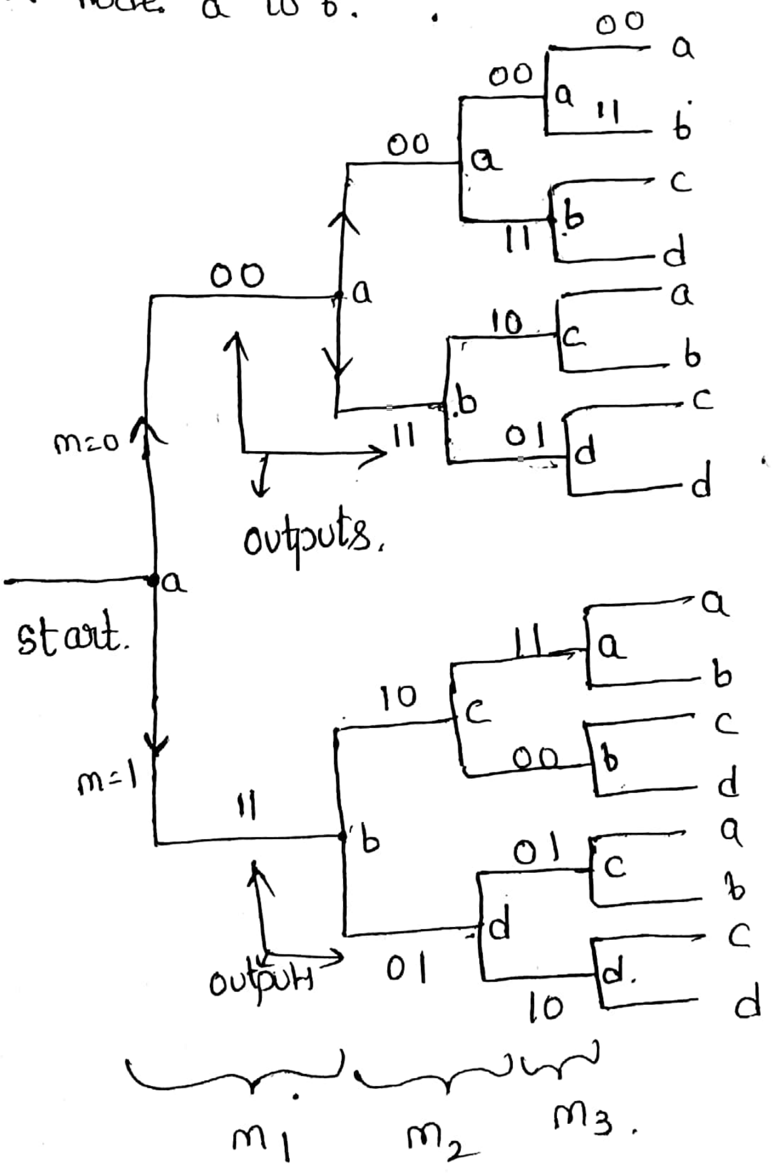
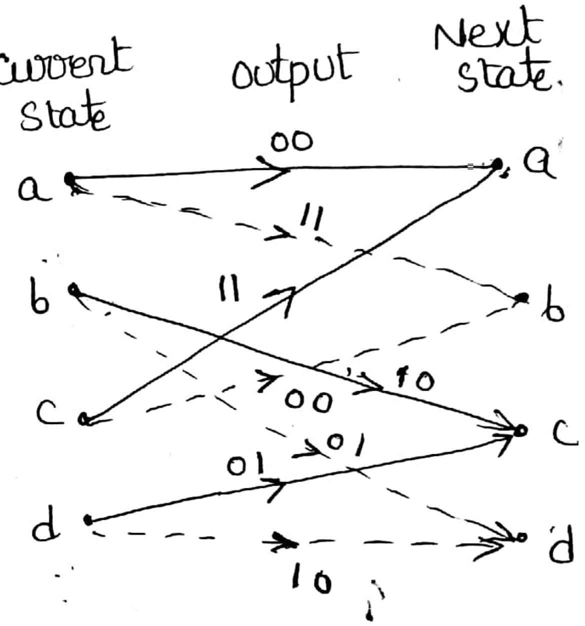


fig :- Code tree

Code trellis of convolutional Encoder :-

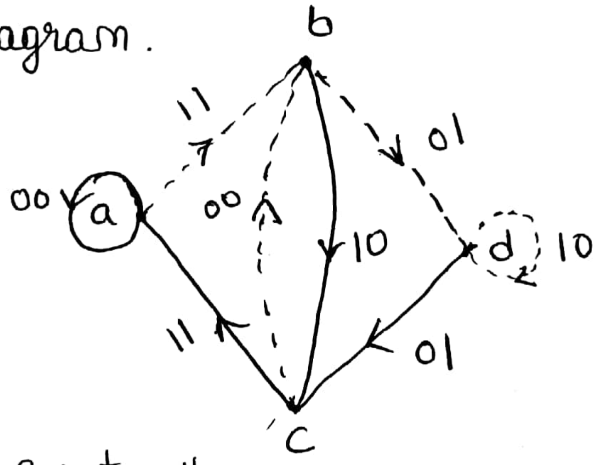
Code trellis is the mōre compact representation the code tree. Every state goes to some other state depending upon the i/p code. Trellis represents the single & unique diagram for such transitions.

The said transition represents input m = 0.
& broken line represents input m = 1.
Along with each transition line the output $x_1 x_2$ is reperesented.



State diagram :-

The current state & next states are combined to obtain state diagram.



1) A rate 1/3 convolution Encoder has generating vectors as $g_1 = (100)$, $g_2 = (111)$ & $g_3 = (101)$

i) sketch the encoder configuration.

ii) Draw the code tree, state diagram & trellis diagram

iii) If i/p message sequence is 10110, determine the output Sequence of the Encoder.

Solution :-

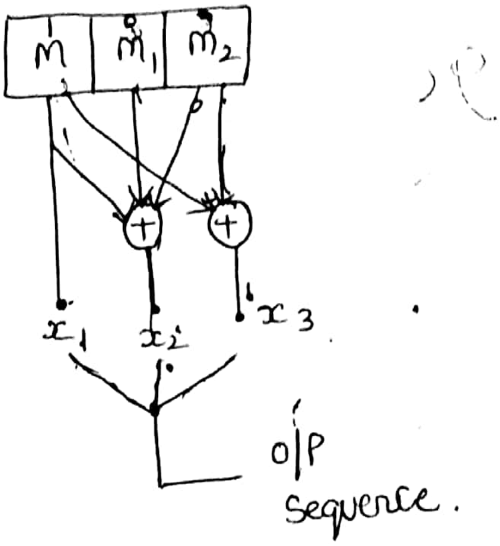
$g_1 = (100) \Rightarrow x_1 = m$

$g_2 = (111) \Rightarrow x_2 = m + m_1 + m_2$

$g_3 = (101) \Rightarrow X_3 = m + m_2$

i) Encoder :-

S.NO	Current state	Input M	outputs X_1 X_2 X_3	Next State
1	a = 00	0	0 0 0	a
		1	1 1 1	b
2.	b = 01	0	0 1 0	c
		1.	1 0 1	d
3.	c = 10	0	0 1 1	a
		1	1 0 0	b
4	d = 11	0	0 0 1	c
		1	1 1 0	d.

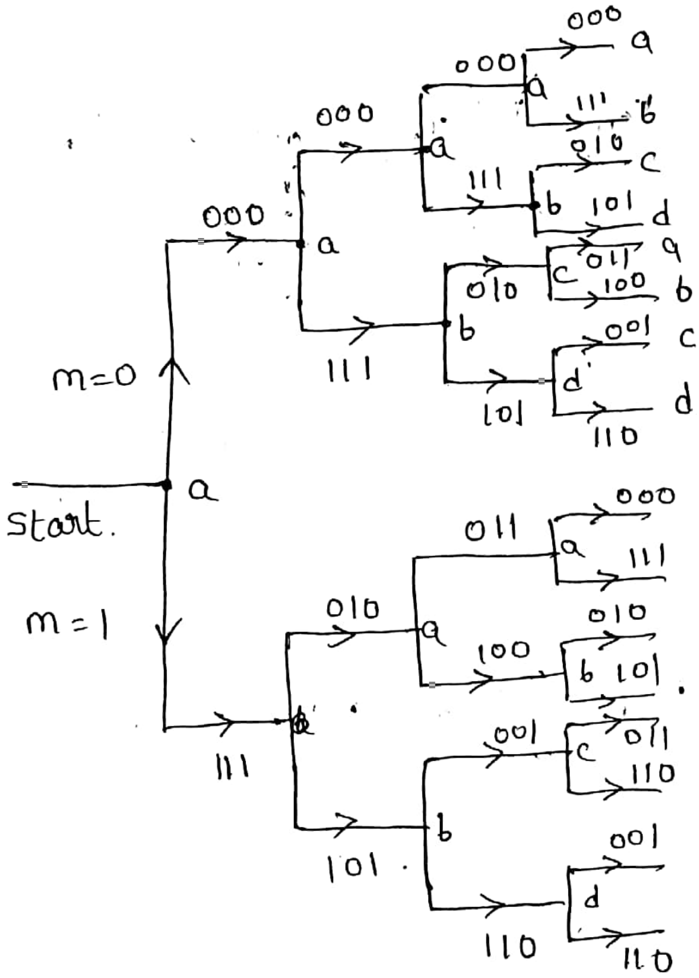


$X_1 = m$

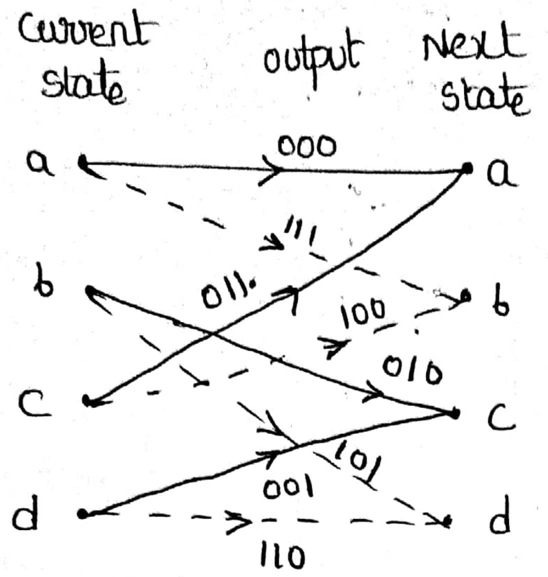
$X_2 = m \oplus m_1 \oplus m_2$

$X_3 = m \oplus m_2$

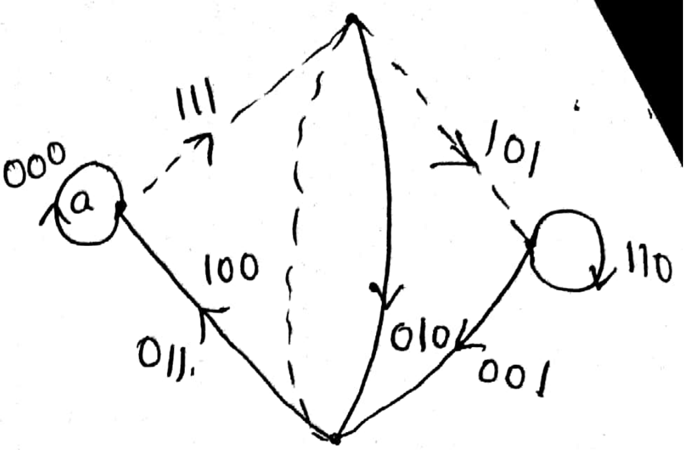
code tree :-



Trellis diagram:-

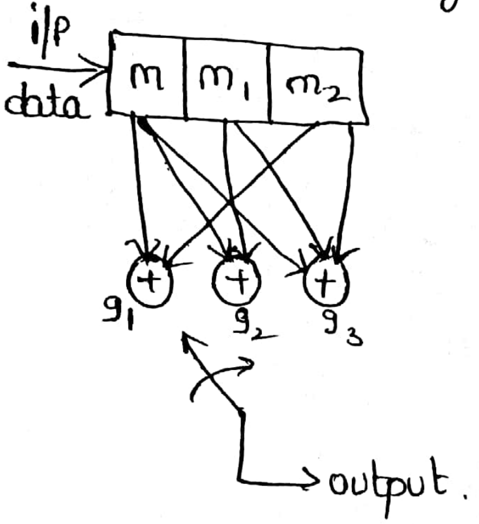


state diagram:-



2. Draw the state diagram, tree diagram, & trellis diagram for K = 3, rate = $\frac{1}{3}$ code generated by $g_1(x) = 1+x^2$, $g_2(x) = 1+x$ & $g_3(x) = 1+x+x^2$

Solution:- Encoder diagram.



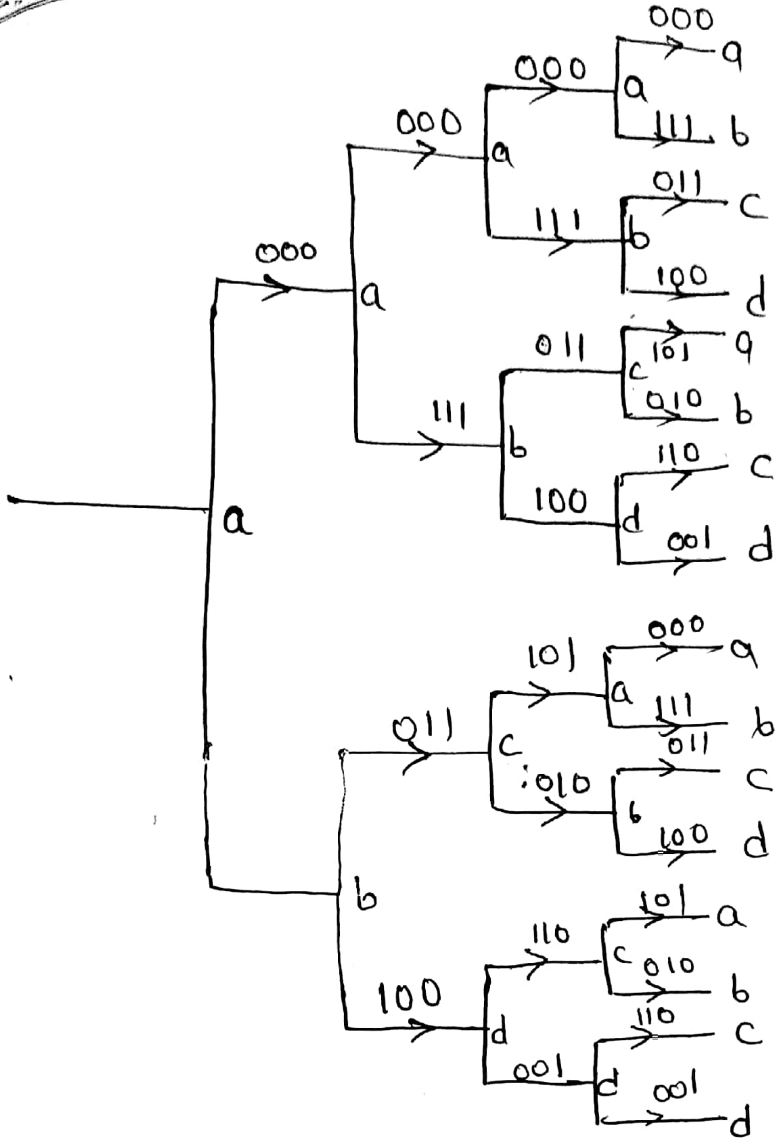
$g_1 = m \oplus m_2$

$g_2 = m \oplus m_1$

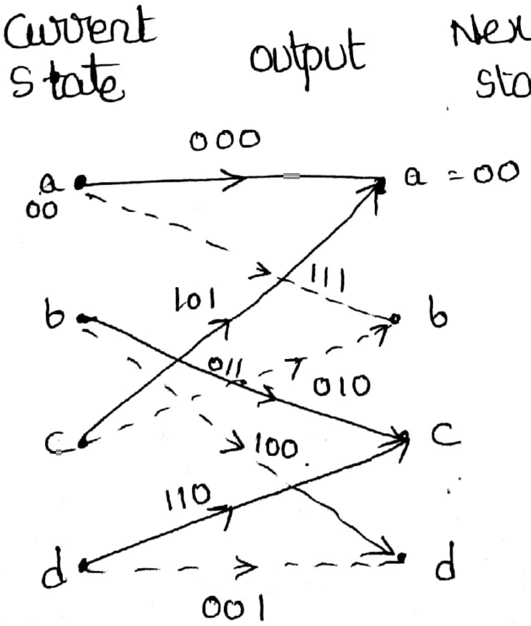
$g_3 = m \oplus m_1 \oplus m_2$

SL No	Current state m_2 m_1	Input m	outputs $g_1 = m \oplus m_2$ $g_2 = m \oplus m_1$ $g_3 = m \oplus m_1 \oplus m_2$	Next State
1	0 0 a	0	0 0 0	a
		1	1 1 1	b
2	0 1 b	0	0 1 1	c
		1	1 0 0	d
3	1 0 c	0	1 0 1	a
		1	0 1 0	b
4	1 1 d	0	1 1 0	c
		1	0 0 1	d.

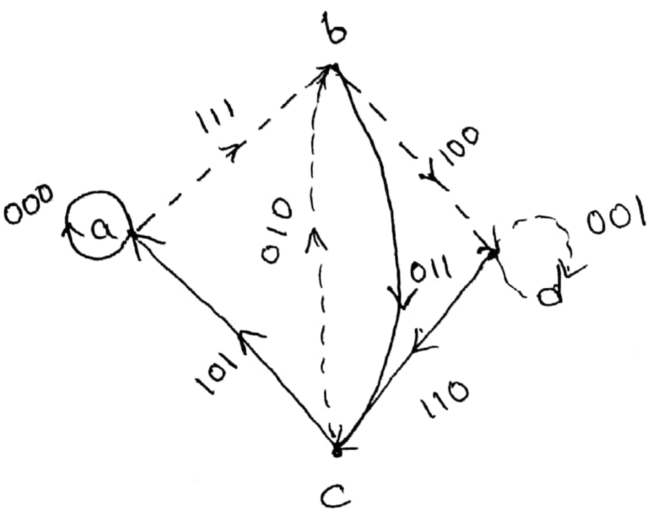
e. Tree :-



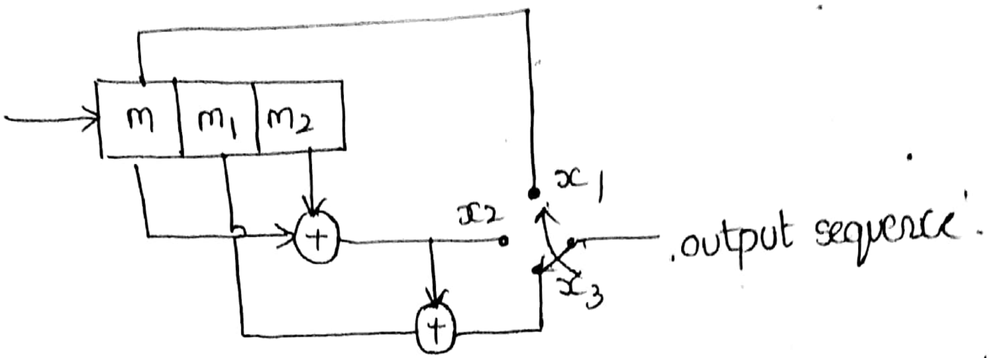
Trellis diagram



state diagram



→ For a convolutional Encoder shown in below figure. Sketch the diagram and trellis diagram. Determine the output data for the input data sequence of 10110..



Solution:- In this k=1 and n=3. This is rate $\frac{1}{3}$ Encoder & constraint length is K=3.

→ To prepare state diagram and trellis diagram :-

$x_1 = m$

$x_2 = m \oplus m_2$

$x_3 = x_2 \oplus m_1$

$x_3 = m_2 \oplus m_1 \oplus m$.

Sr. NO	Current state m_2 m_1	input m	outputs. $x_1 = m$ $x_2 = m \oplus m_2$ $x_3 = m \oplus m_1 \oplus m_2$	Next state m_1 m_2
1	a = 00	0	0 0 0	00 (a)
		1	1 1 1	01 (b)
2.	b = 01	0	0 0 1	10 (c)
		1	1 1 0	11 (d).
3.	c = 10	0	0 1 1	00 (a)
		1	1 0 0	01 (b)
4.	d = 11	0	0 1 0	10 (c)
		1	1 0 1	11 (d).

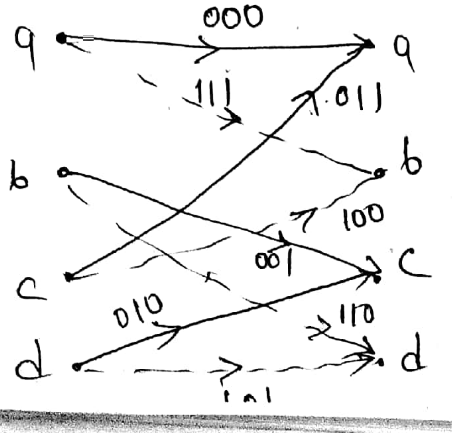
a = 00

b = 01

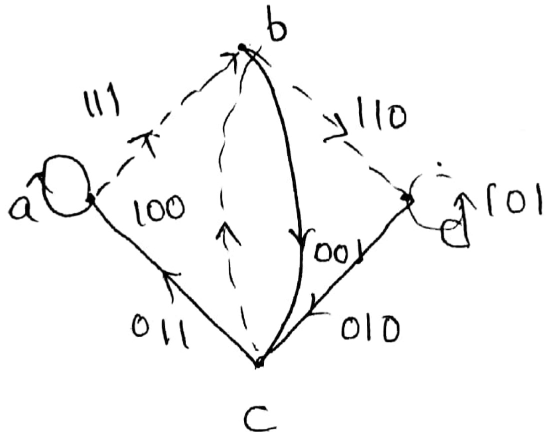
c = 10

d = 11.

Trellis diagram :-



State diagram :-



btain output sequence x_1, x_2 and x_3

$m = 10110$

0 1 2 3 4

$m(P) = 1+p^2+p^3.$

$x_1 = 1 \quad x_1(P) = 1$

$x_2 = 101 \quad x_2(P) = 1+p^2$

$x_3 = 111 \quad x_3(P) = 1+P+P^2$

$x_i^{(1)} = x_1(P).m(P)$

$= 1.(1+p^2+p^3)$

$= 1+p^2+p^3.$

$x_1 = (1011000)$

$x_2 = x_2(P).m(P)$

$= (1+p^2).(1+p^2+p^3)$

$= 1+\cancel{p^2}+p^3+\cancel{p^2}+p^4+p^5$

$= 1+p^3+p^4+p^5$

$x_2 = (1001110)$

$x_3 = x_3(P).m(P)$

$= (1+P+P^2).(1+P^2+P^3)$

$= 1+\cancel{p^2}+\cancel{p^3}+P+\cancel{p^3}+\cancel{p^4}+\cancel{p^2}+\cancel{p^4}+p^5$

$= 1+P+p^5$ multiplexed above sequences,

$x_3 = \{110001\}$

$x_i = \{111\ 001\ 100\ 110\ 010\ 011\}.$

Decoding methods of convolutional codes:-

veterbi algorithm, sequential decoding and feedback decoding are used for decoding for convolutional codes.

Viterbi Algorithm for Decoding of convolutional codes :-

(maximum likelihood Decoding).

The viterbi algorithm operates on the principle of maximum likelihood decoding and achieves optimum performance. The maximum likelihood decoder has to examine the entire received sequence Y and find a valid path which has the smallest hamming distance from Y. But there are 2^N possible paths for a message sequence of N bits. There are a large number of paths.

Metric :-

It is defined as the Hamming distance of each bran each surviving path from the corresponding branch of Y The metric is defined by assuming that 0's and 1's have the same transmission error probability.

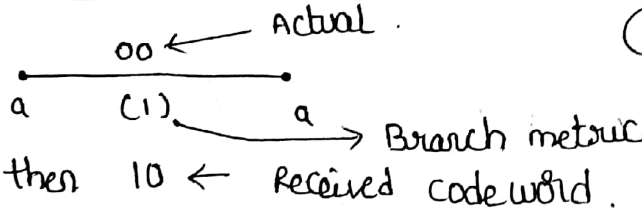
Surviving path :- It is the path of the decoded signal with minimum metric.

i) let the received signal be represented by Y. The viterbi decoder assigns to each branch of each surviving path of a metric

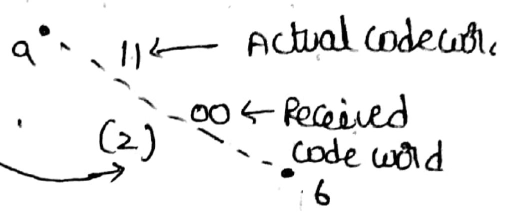
ii) By Summing the branch metrices we get the path metric

Branch metric :- → The diff of Actual and received code word.

Eg :- 1)

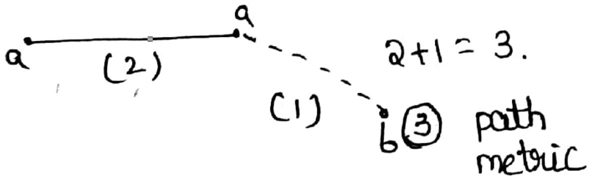


②

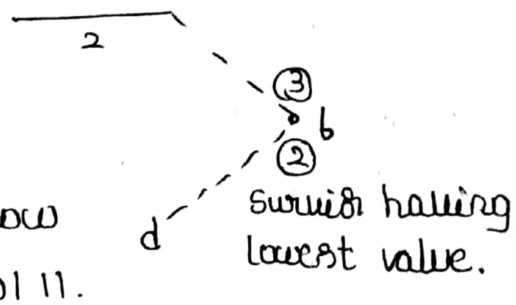


path metric :-

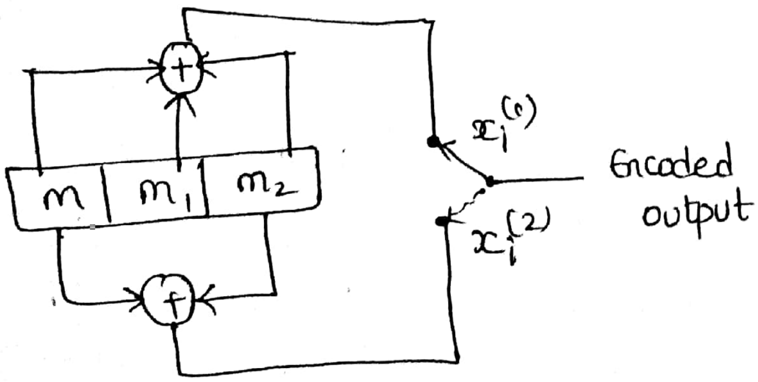
Eg :-



survivor path :-

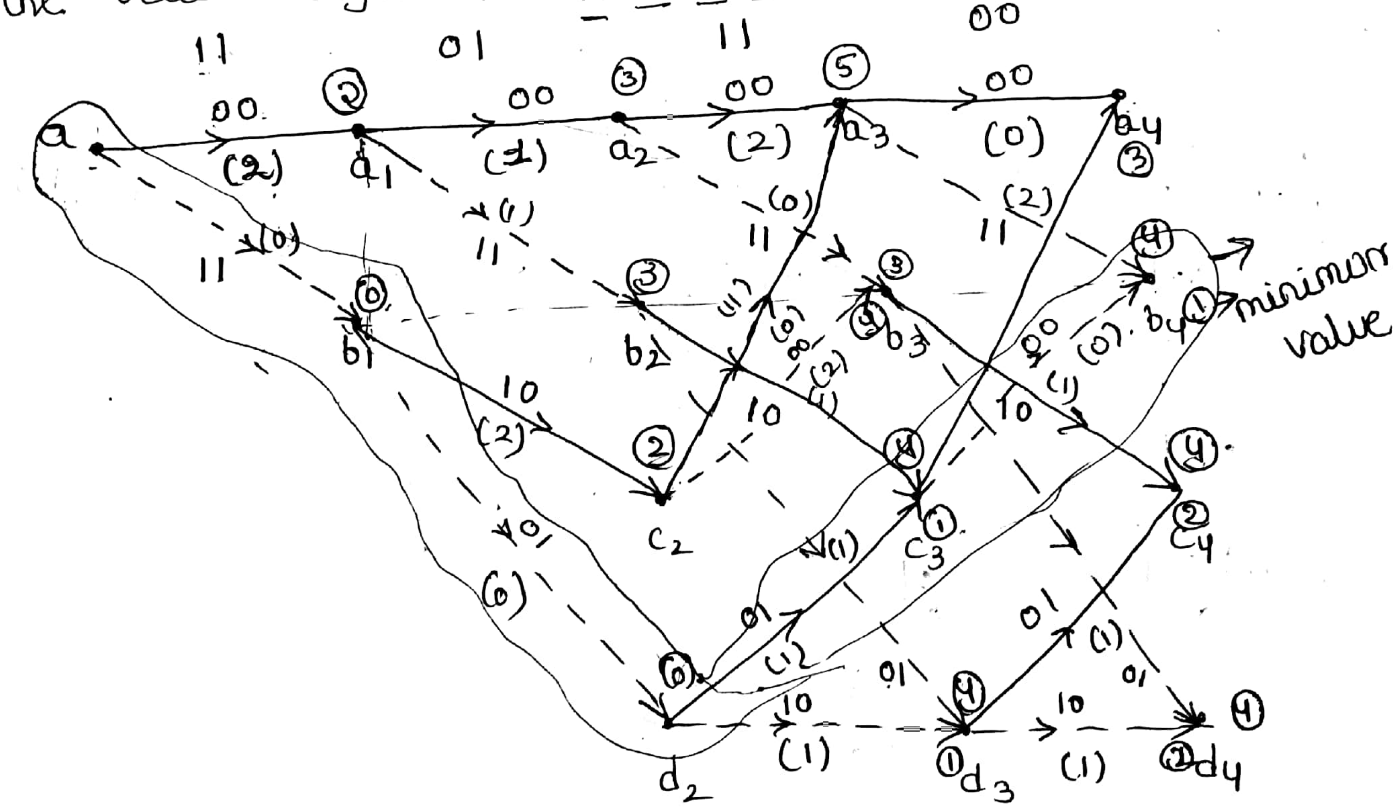


1) Given the convolutional encoder in below figure and for a received signal Y = 110111.
Show that the first three branches of the valid paths emerging from the initial node a0 in the code trellis.



urrent state	Input message bit m	Shift Register m m₁ m₂	output x₁ x₂	Next state m₂ m₁	State
a	0	0 0 0 0 0	0 0	0 0	a
(00)	1	1 0 0 1 0	1 1	0 1	b
b	0	0 1 0 0 1	1 0	1 0	c
(01)	1	1 1 0 1 0	0 1	0 1	d
c	0	0 0 1 0 0	1 1	0 0	a
(10)	1	1 0 1 1 0	0 0	0 1	b
d	0	0 1 1 0 1	0 1	1 0	c
(11)	1	1 1 1 1 1	1 0	1 1	d

The following state transition calculation is considered with the received signal Y = 11 01 11 00



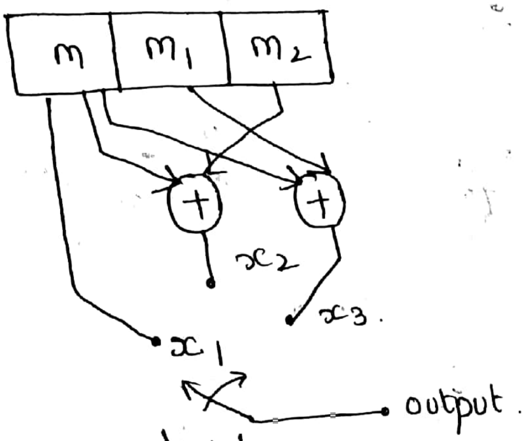
Y + E = 11 01 01 00

m = 1 1 0 1.

S.NO	path	path metric	Decision
1	$a_0 a_1 a_2 a_3$	5	X
2.	$a_0 a_1 a_2 b_3$ (survivor)	3	✓
3.	$a_0 a_1 b_2 c_3$	4	X
4.	$a_0 a_1 b_2 d_3$	4	X
5.	$a_0 b_1 c_2 a_3$ (survivor)	2	✓
6	$a_0 b_1 c_2 b_3$	4	X
7	$a_0 b_1 d_2 c_3$ (survivor)	1	✓
8	$a_0 b_1 d_2 d_3$ (survivor)	1	✓

2) For the convolutional encoder arrangement shown in figure. what are the dimensions of the code (n,k) and constraint length. use viterbi's algorithm to decode the sequence.

100 110 111 101 001 101 001 010.



solution:-

Dimensions of the code (n,k) = (3,1)

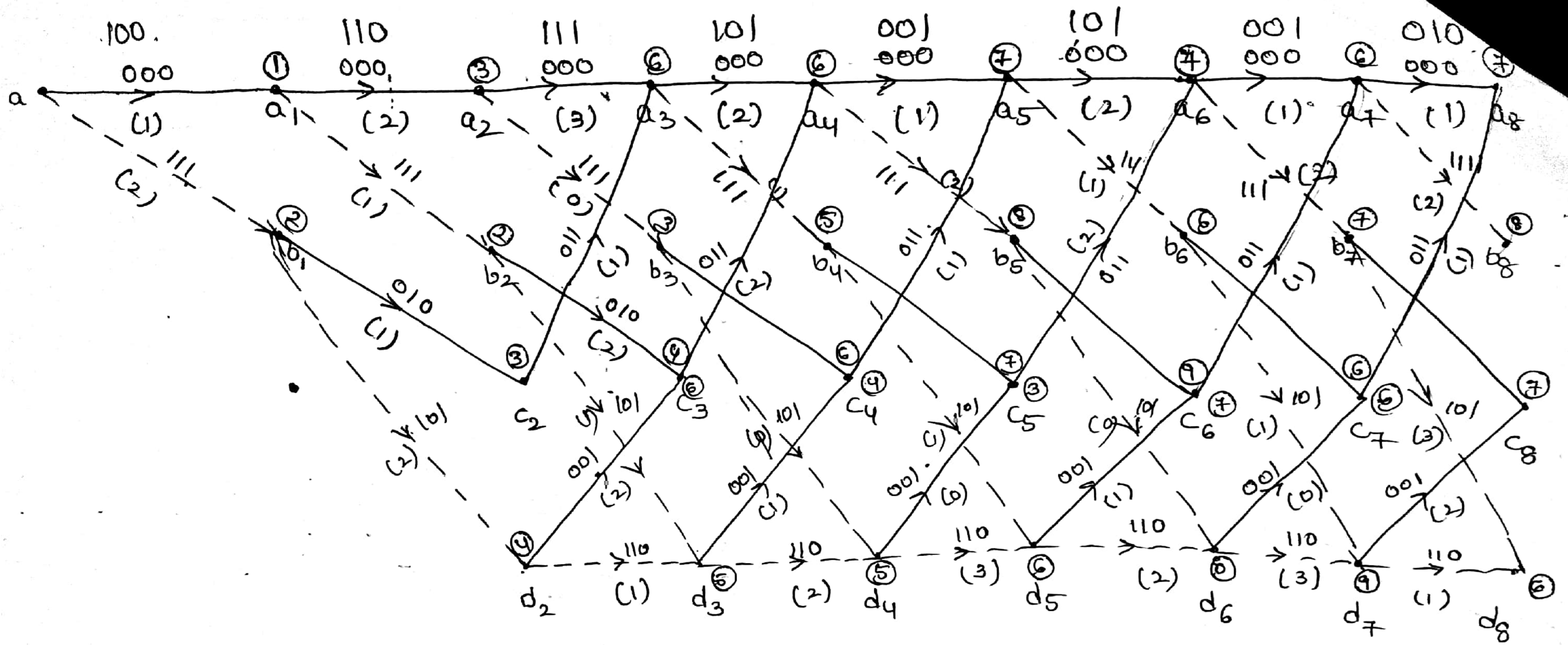
constraint length (K) = 3.

$x_1 = m$

$x_2 = m \oplus m_2$

$x_3 = m \oplus m_1$.

current state.	m	o/p	next state
a	0	000	→ a
	1	111	→ b.
b	0	010	– c
	1	101	– d
c	0	011	→ a
	1	100	– b
d	0	001	– c
	1	110	– d



minimum distance value = 6.

Y + E = 000 000 111 101 001 100 101 110

m = 0 0 1 1 0 1 1 1.

→ A rate $\frac{1}{2}$ convolution encoder has generating vectors

and $g_2 = \{1,1,0\}$

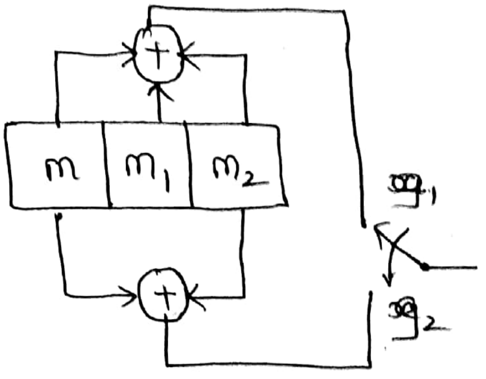
a) sketch the encoder configuration.

b) find constrain length, code dimension.

c) draw code tree, state tree and trellis diagram.

d) If input is 10110. Determine output sequence of encoder

e) If received signal is 101100101101. decode the input signal.



Solution :-

a) Encoder configuration

$g_1 = m \oplus m_1 \oplus m_2$

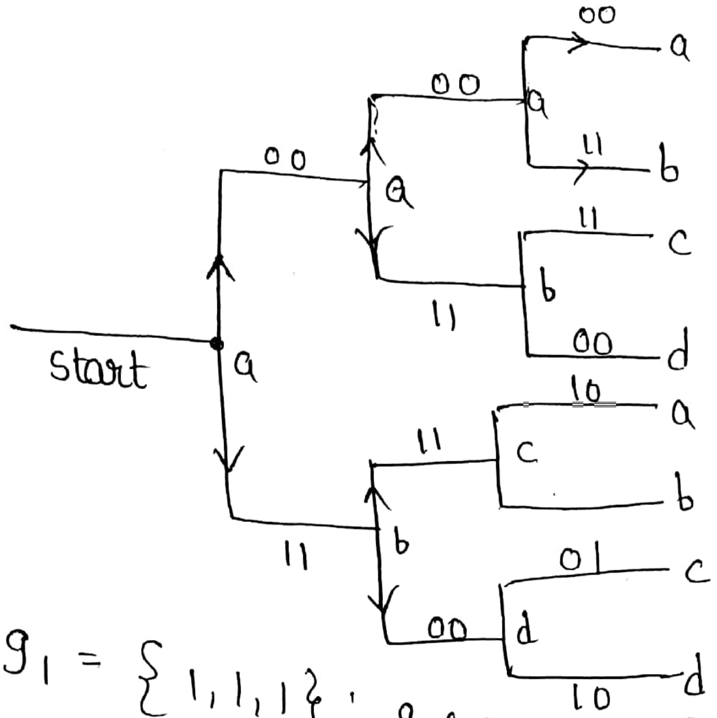
$g_2 = m \oplus m_1$

b) Constrain length $k = 3$, dimensions of the code $(n, k) = (2,1)$

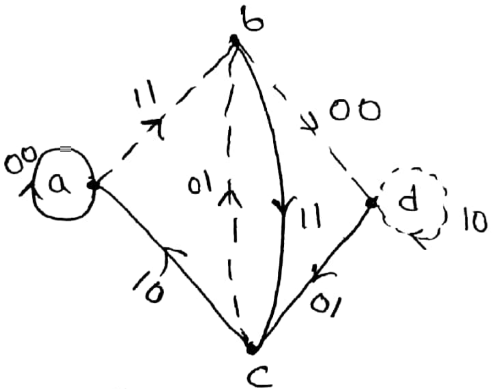
c) Code tree :-

Initial state	input bit m	contents of shift register	output g_1 g_2	next state
a	0	0 0 0	0 0	a
	1	1 0 0	1 1	b
b	0	0 1 0	1 1	c
	1	1 1 0	0 0	d
c	0	0 0 1	1 0	a
	1	1 0 1	0 1	b
d	0	0 1 1	0 1	c
	1	1 1 1	1 0	d

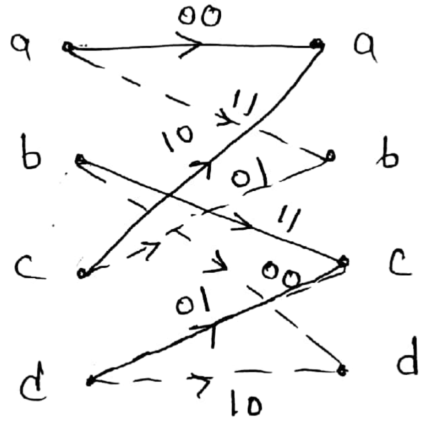
de tree :-



State diagram



Trellis diagram



d) $g_1 = \{1,1,1\}$; $g_1(P) = 1+P+P^2$

$g_2 = \{1,1,0\}$, $g_2(P) = 1+P$

$m = 10110$, $m(P) = (1+P^2+P^3)$

adder 1 output $x_i^{(1)} = g_1(P).m(P)$

$$= (1+P^2+P)(1+P^2+P^3)$$

$$= 1+\cancel{P^2}+\cancel{P^3}+\cancel{P^2}+\cancel{P^4}+P^5+P+\cancel{P^3}+\cancel{P^4}$$

$$= 1+P+P^5 = (1100010)$$

adder 2 output $x_i^{(2)} = g_2(P).\,m(P)$

$$= (1+P)(1+P^2+P^3)$$

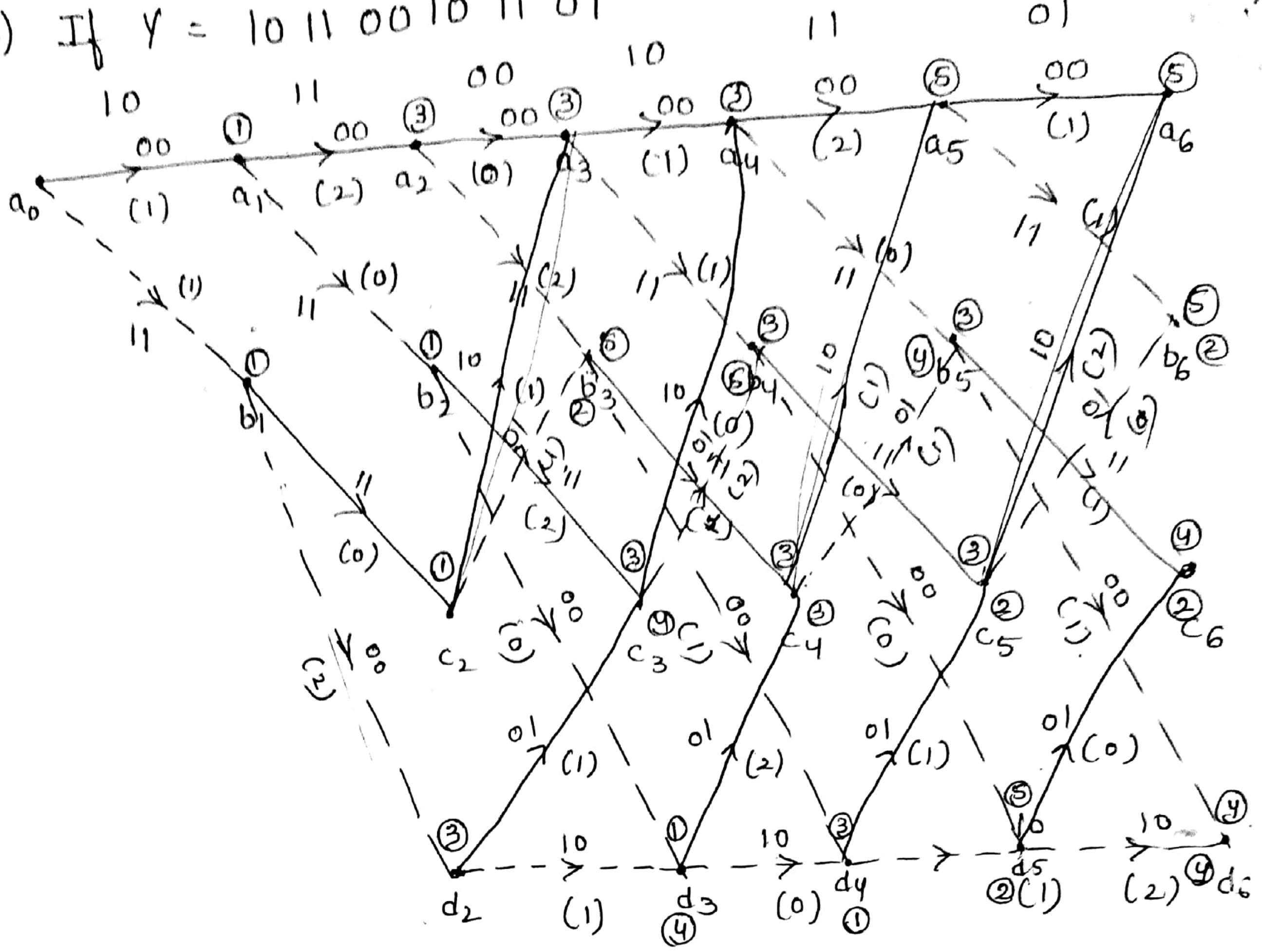
$$= 1+P^2+\cancel{P^3}+P+\cancel{P^3}+P^4$$

$$= 1+P+P^2+P^4$$

$$= (1110100)$$

$x_i = \{11, 11, 01, 00, 01, 10, 00\}$.

e) If Y = 10 11 00 10 11 01



$Y + E$ = 00 11 00 10 01 01

m = 0 1 1 1 0 1

→ using viterbi algorithm, if the received signal at the decoder Y = 00 01 10 00 00 00 10 01. find transmitted sequence.

⇛

initial state	message bit	output	next state
a	0	00	a
	1	11	b
b	0	10	c
	1	01	d
c	0	11	a
	1	00	b
d	0	01	c
	1	10	d.